Depths of Field & Focus

Depth of Field:

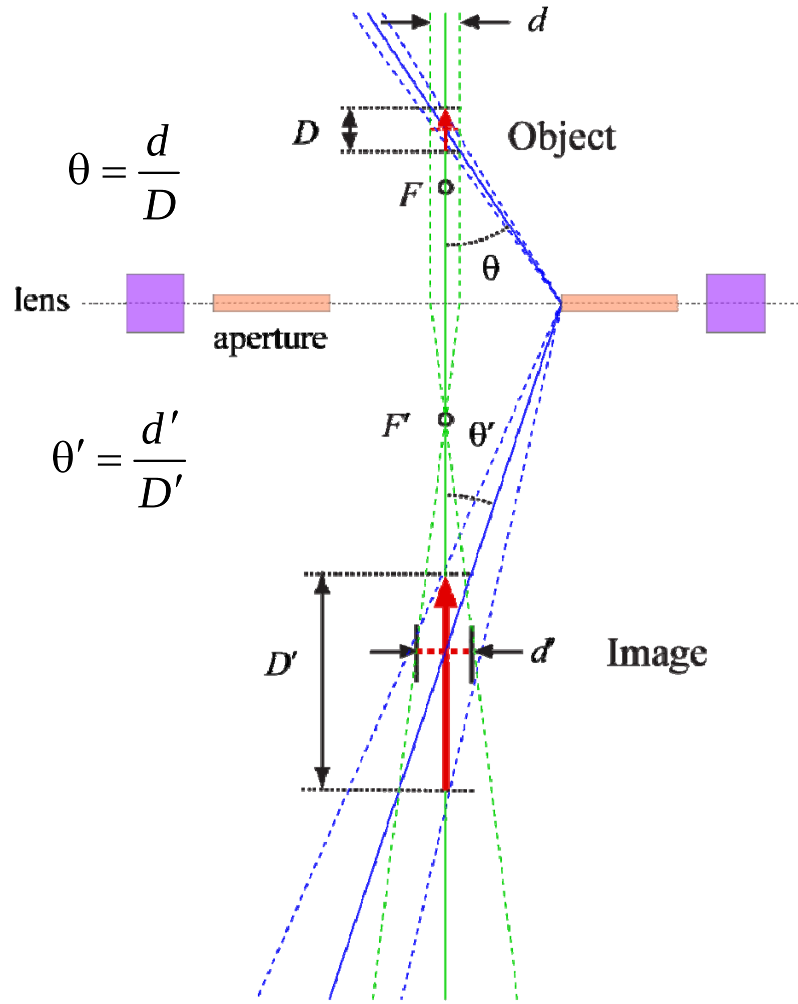
If the resolution is δ, then
the entire object is in focus when:

$$d < \delta \qquad\qquad D < \frac{\delta}{\theta}$$

$$\text{depth of field} = \frac{\delta}{\theta}$$

$$\theta = \frac{d}{D}$$

$$\theta' = \frac{d'}{D'}$$



Depth of Focus:

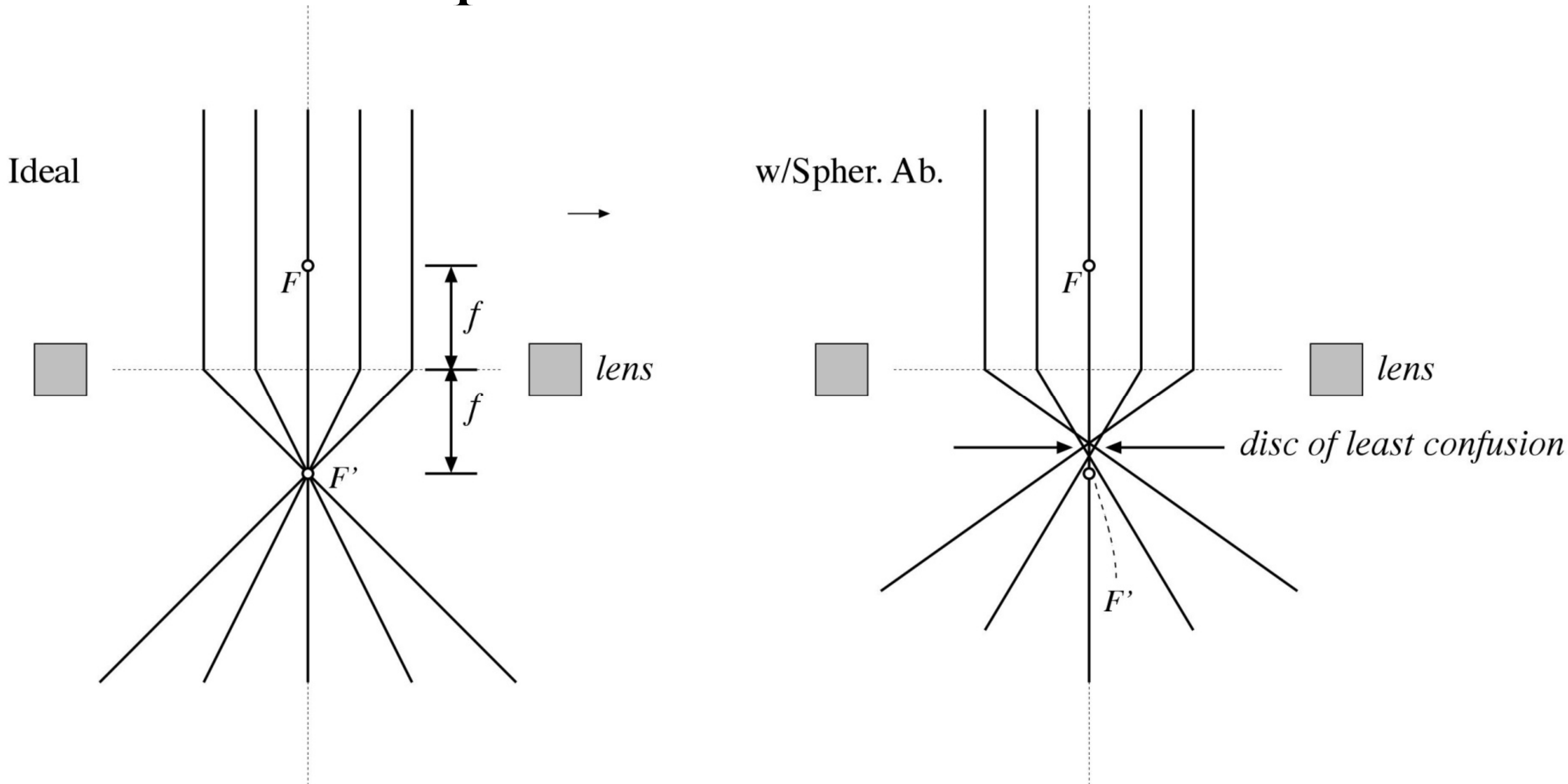
Image in focus if:

$$d' < M\delta \qquad D' < \frac{M\delta}{\theta'} \qquad M_\theta = \frac{\theta'}{\theta} = \frac{1}{M}$$

$$D' < \frac{M\delta}{\theta'}\left(= M \cdot \frac{\theta}{\theta'} \cdot \frac{\delta}{\theta} = M^2 \cdot \frac{\delta}{\theta} \right)$$

$$\text{depth of focus} = M^2 \cdot \text{depth of field}$$

Spherical aberration



High-angle rays focused more strongly

An ideal lens is parabolic, not spherical

Parabolic approximation of a sphere

1) Sphere:

$$y - R = \sqrt{R^2 - x^2}$$

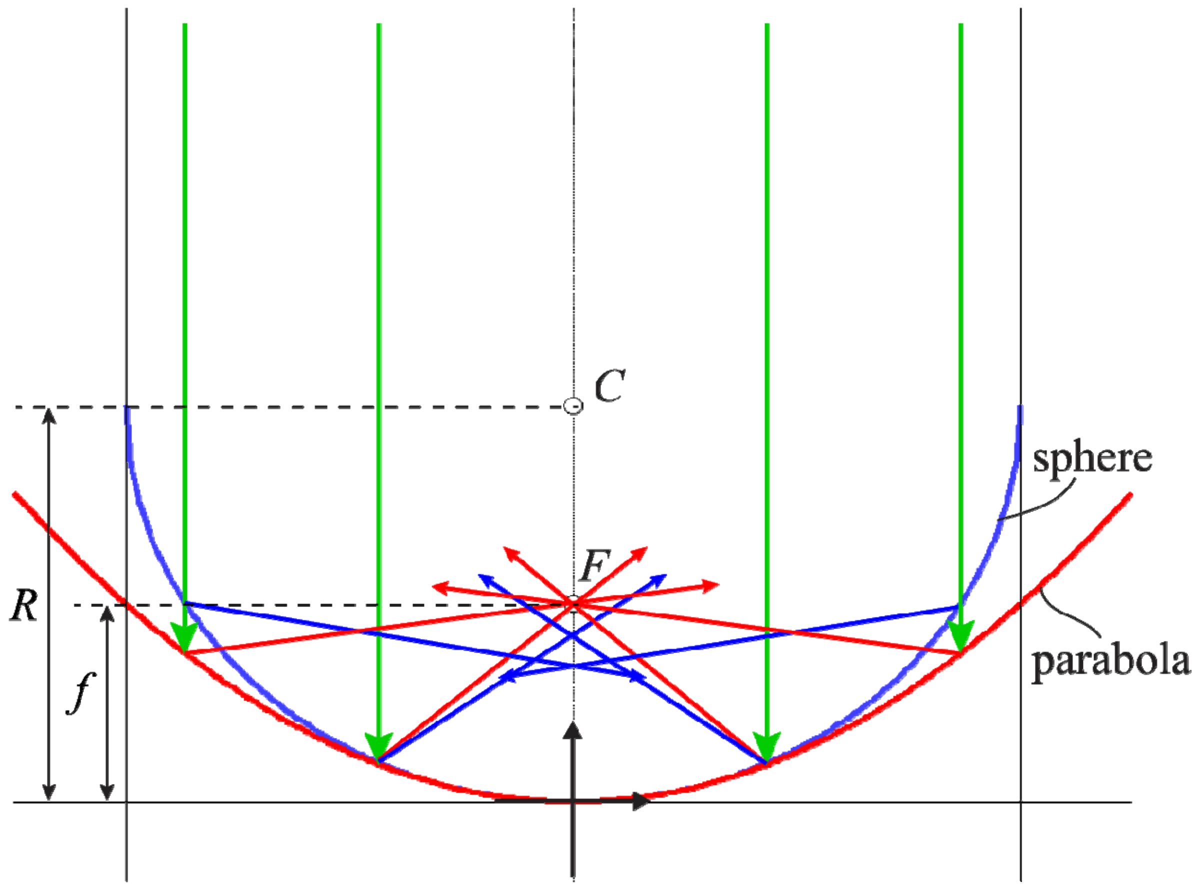
Small angle: $y \approx \dfrac{x^2}{2R}$

2) Parabola:

$$y + f = \sqrt{(y - f)^2 + x^2}$$

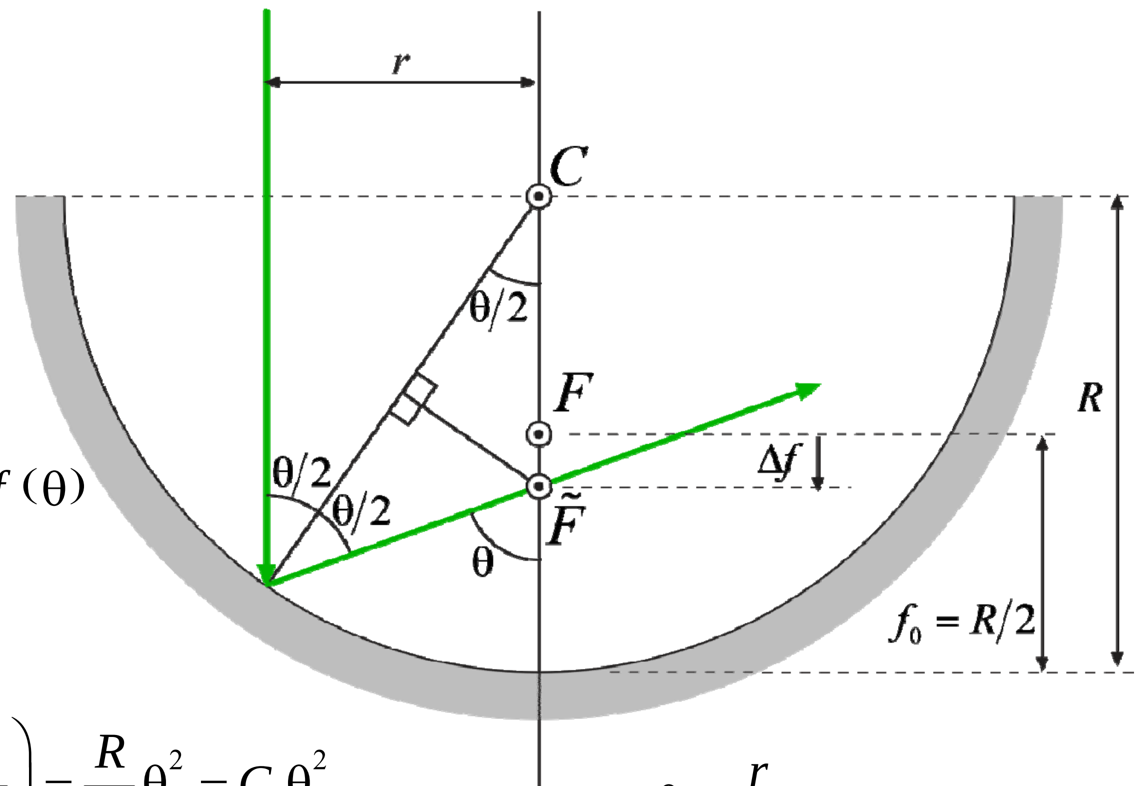
or: $y = \dfrac{x^2}{4f}$

$\Rightarrow \quad f = R/2$



Origin of spherical aberration

Find C_s for a sphere:



$$f(\theta) = \frac{R}{2}\left(2 - \frac{1}{\cos(\theta/2)}\right) = f_0 - \Delta f(\theta)$$

$$f_0 = \frac{R}{2}$$

$$\Delta f(\theta) = \frac{R}{2}\left(\frac{1}{\cos(\theta/2)} - 1\right) \approx \frac{R}{2}\left(\frac{\theta^2}{8}\right) = \frac{R}{16}\theta^2 = C_s\theta^2$$

$$\theta \approx \frac{r}{f_0}$$

$$C_s = \frac{R}{16} = \frac{f_0}{8}$$

$$f(\theta) = f_0 - C_s\theta^2$$

$$\Delta f(r) = C_s\left(\frac{r}{f_0}\right)^2$$

Effect of Δf in Image Plane

$\theta \to 0:$ $$\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{f_0}$$

Otherwise: $$\frac{1}{p_0} + \frac{1}{q_0 + \Delta q} = \frac{1}{f_0 - \Delta f}$$

Expand: $$\frac{1}{1-x} = 1 + x^1 + x^2 + \ldots$$

$|\Delta f| \ll f_0$ $$-\frac{\Delta q}{{q_0}^2} \approx \frac{\Delta f}{{f_0}^2}$$

$$\Delta q = -\left(\frac{q_0}{f_0}\right)^2 \cdot \Delta f$$

Effect of spherical aberration on resolution

Ray misses crossover at Gaussian image plane $\Delta f \approx C_s\theta^2$

C_s : spherical aberration coefficient (typically 1-3 mm)

High mag:

$$M = \frac{q_0}{p_0} = \frac{q_0}{f_0} - 1 \approx \frac{q_0}{f_0}$$

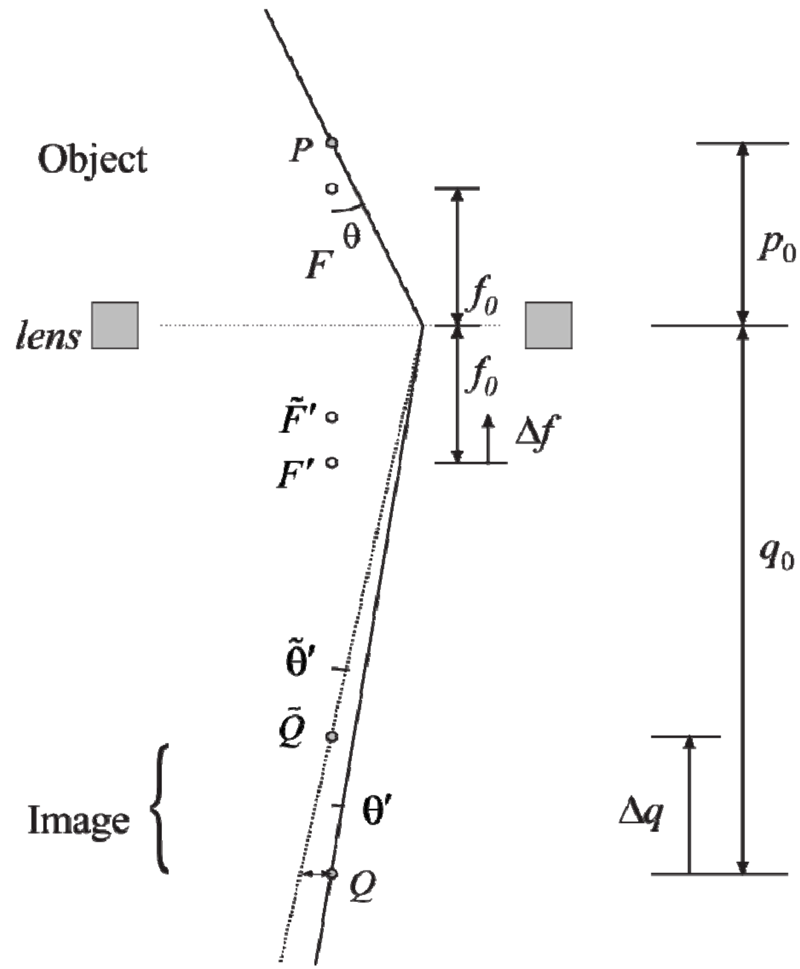
$$\Delta q = -\left(\frac{q_0}{f_0}\right)^2 \cdot \Delta f \approx -M^2 C_s \theta^2$$

Image:

$$\tan\theta' = -\frac{\delta'}{\Delta q} \approx \theta' = \frac{\theta}{M}$$

$$\delta' \approx -\theta' \cdot \Delta q = M C_s \theta^3$$

Object: $$\delta = \frac{\delta'}{M} = C_s \theta^3$$



Optimal β

Diffraction

$$\delta_d = (0.61)\frac{\lambda}{\beta}$$

Spherical Aberration

$$\delta_s = C_s \cdot \beta^3$$

Define:

$$\delta_0 \equiv \left(C_s \lambda^3\right)^{1/4} \qquad \beta_0 \equiv \left(\frac{\lambda}{C_s}\right)^{1/4}$$

Rewrite:

$$\delta_d = (0.61) \times \delta_0 \times \left(\frac{\beta_0}{\beta}\right)$$

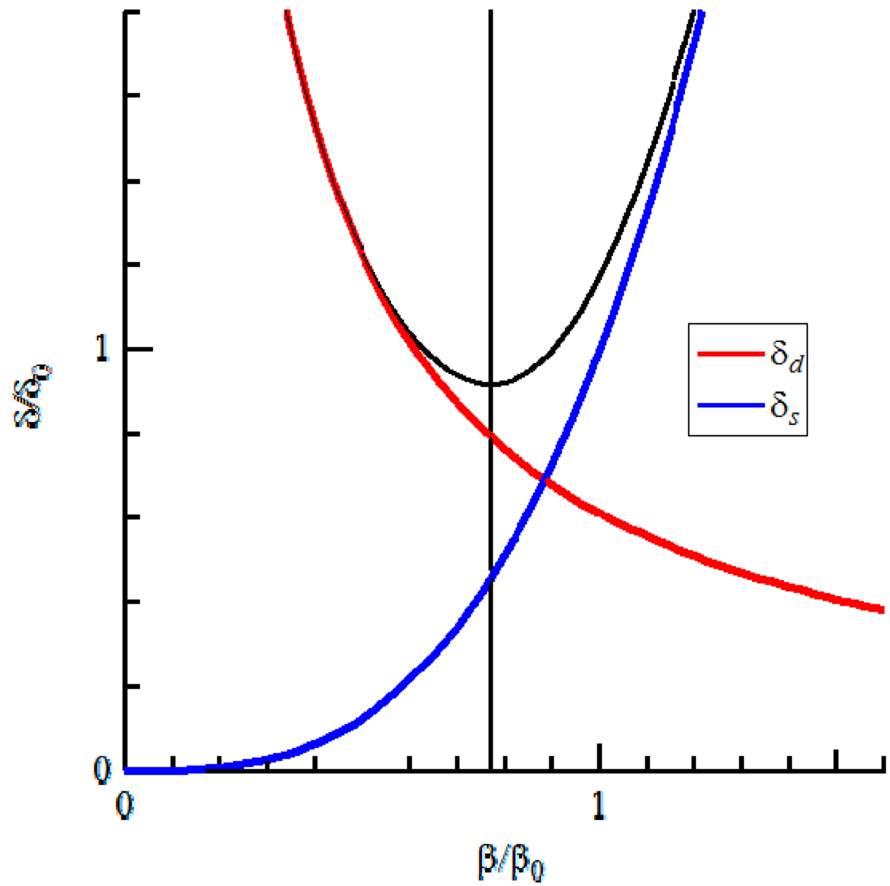
$$\delta_s = \delta_0 \times \left(\frac{\beta}{\beta_0}\right)^3$$

Combine:

$$\delta_{net} \approx \sqrt{\delta_d{}^2 + \delta_s{}^2}$$

Minimize:

$$\left.\frac{d\delta_{net}}{d\beta}\right|_{\beta=\beta_{opt}} = 0 \quad \Rightarrow$$

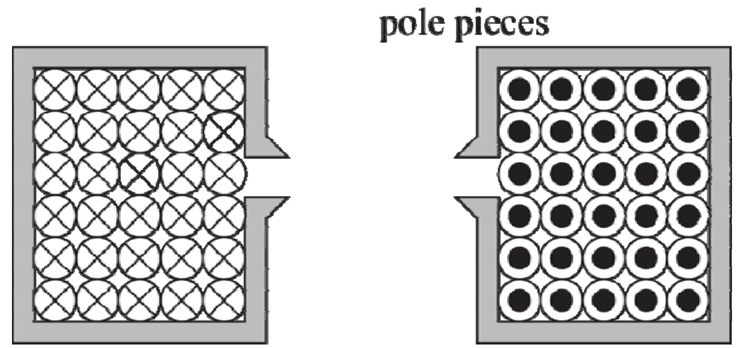


$$\beta_{opt} = \frac{(0.61)^{1/4}}{3^{1/8}} \cdot \beta_0 = (0.77) \cdot \beta_0$$ optimal semi-angle of collection

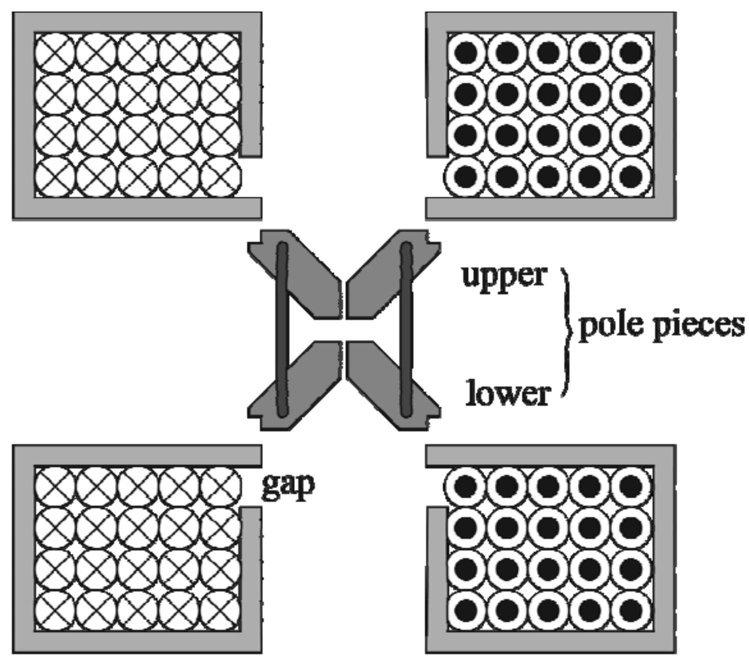
$$\delta_{min} = (0.61)^{3/4} \cdot \sqrt{3^{1/4} + 3^{-3/4}} \cdot \delta_0 = (0.91) \cdot \delta_0$$ practical resolution

Electromagnetic lenses

Current-carrying coils
Enclosed in iron
Bore
Gap
Pole Pieces
Water-cooled

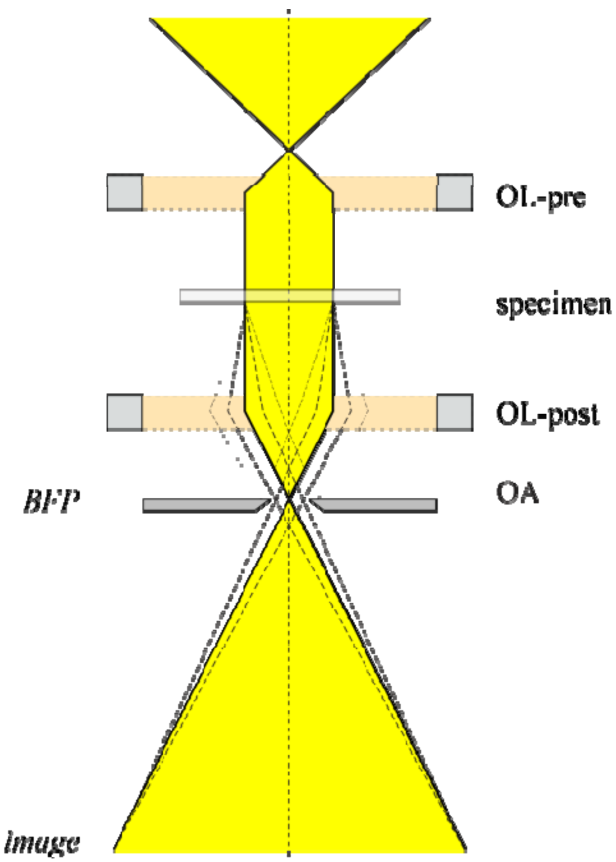


Twin lens

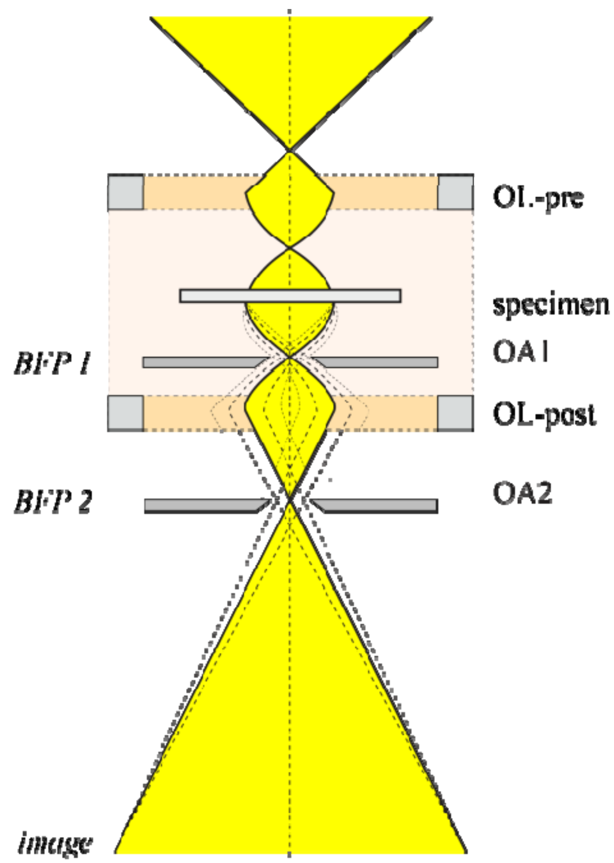


Immersion Lens

idealized
objective lens



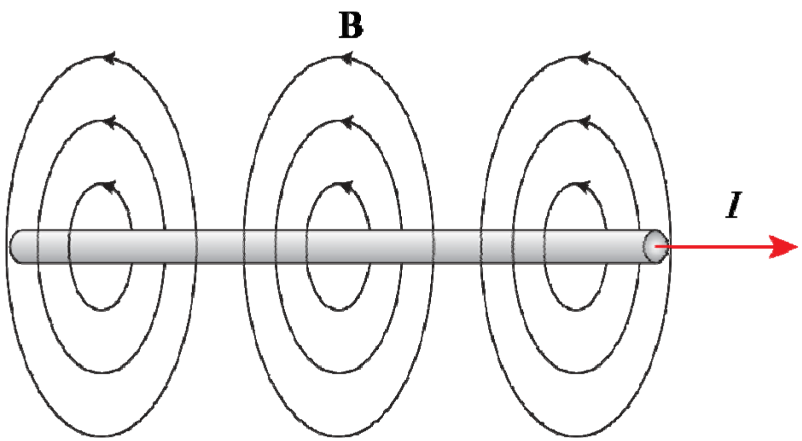
immersion
objective lens



sample in pole piece

Magnetic fields and forces

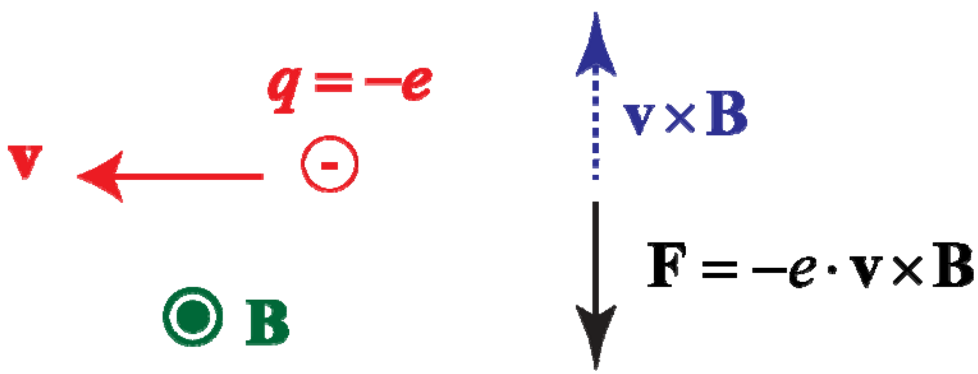
Magnetic fields are caused by electrical currents.



Ampère's law:

$$\oint \mathbf{B} \cdot ds = \mu_0 \cdot I$$

Moving charges in magnetic fields experience forces.



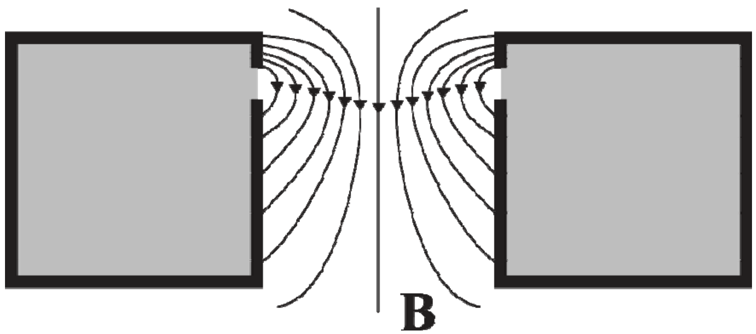
Lorentz force law:

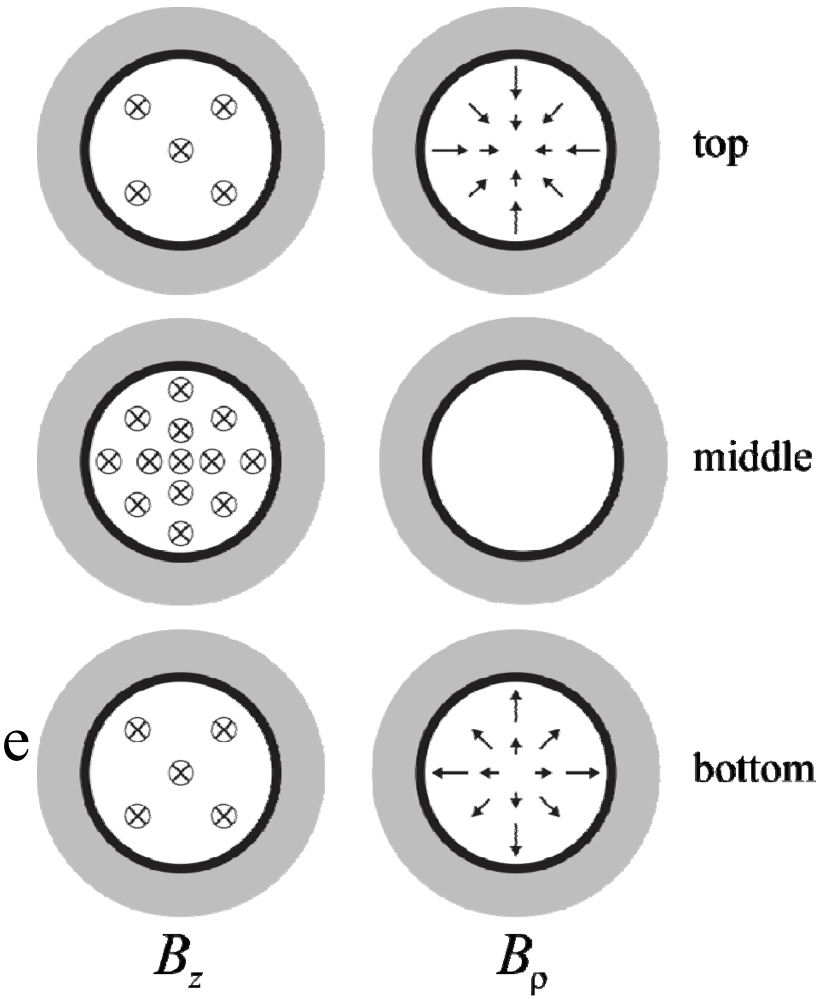
$$\mathbf{F} = q \cdot \mathbf{v} \times \mathbf{B}$$

cross-product

$$|\mathbf{v} \times \mathbf{B}| = v \cdot B \cdot \sin\theta$$

Lens field





axial and radial components
axial symmetry
strongest field on axis, near pole piece
radial component reverses from top
to bottom

Conditions on magnetic field

No magnetic monopoles:

$$\vec{\nabla} \cdot \mathbf{B} = 0$$

$$\vec{\nabla} \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho B_\rho\right) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

//divergence in cylindrical coord's.

$$B_\phi = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho B_\rho\right) = -\rho \frac{\partial B_z}{\partial z}$$

//Axial symmetry

$$\int_\rho \frac{\partial}{\partial \rho}\left(\rho B_\rho\right) \cdot d\rho = -\int_\rho \rho \cdot \left(\frac{\partial B_z}{\partial z}\right) \cdot d\rho$$

//Integrate

Assume:

$$\frac{\partial B_z}{\partial z}(\rho, z) = \frac{\partial B_z}{\partial z}(z)$$

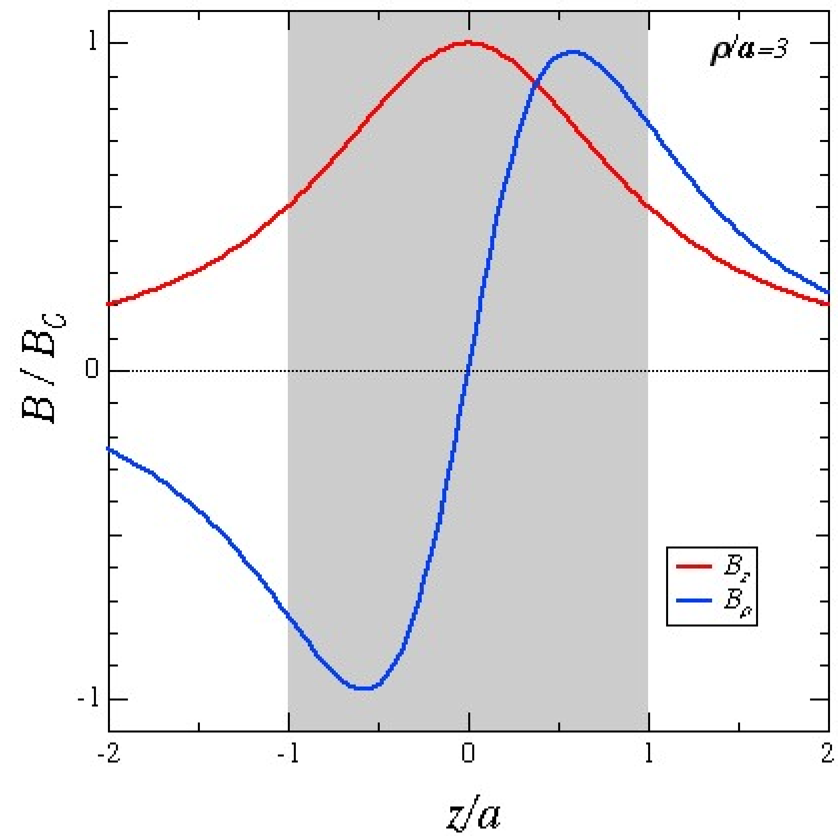
$$\rho B_\rho = -\frac{\rho^2}{2}\frac{\partial B_z}{\partial z}$$

$$B_\rho = -\frac{\rho}{2}\frac{\partial B_z}{\partial z}$$

Model: Bell-shaped field ("Glockenfeld")

Model of lens field:

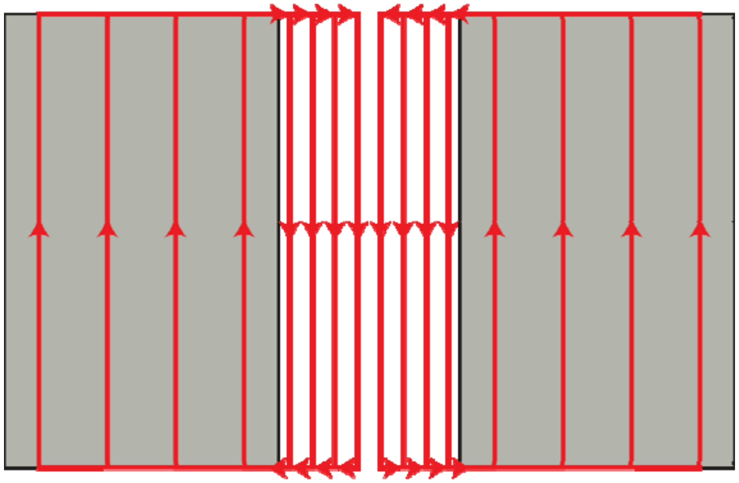
$$B_z(z) = \frac{B_0}{1 + \left(\frac{z}{a}\right)^2}$$

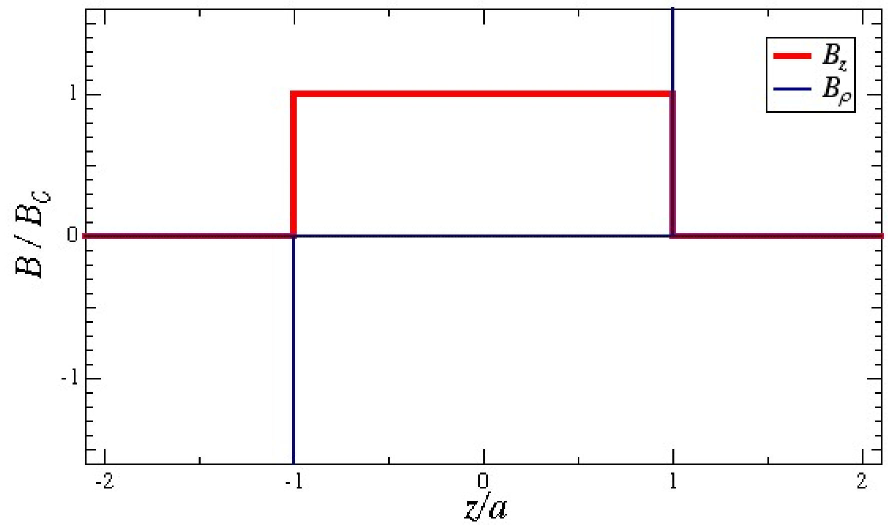


Determine radial component:

$$B_\rho(\rho, z) = \frac{\rho z}{a^2} \cdot \frac{B_z(z)}{1 + \left(\frac{z}{a}\right)^2}$$

Model: Uniform B_z in lens



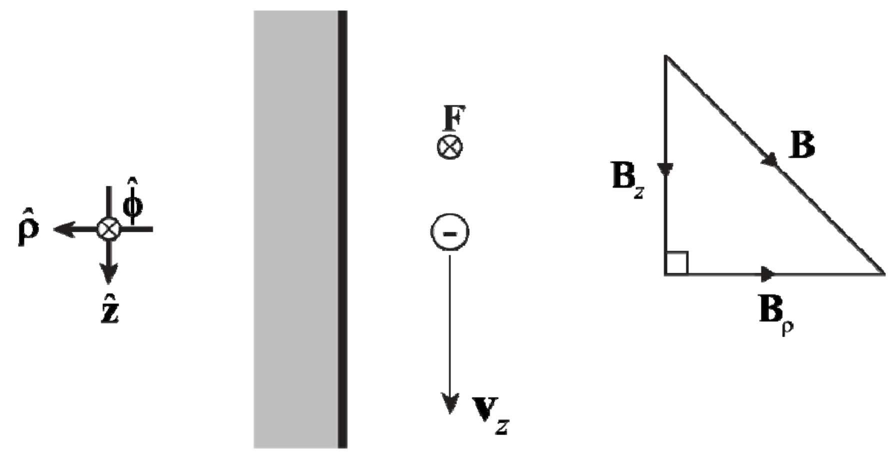


$$B_z = B_0 \cdot \left[u(z+a) - u(z-a) \right]$$

$$B_\rho = -\frac{\rho B_0}{2} \cdot \delta(z+a) + \frac{\rho B_0}{2} \delta(z-a)$$

Focusing action

Side View

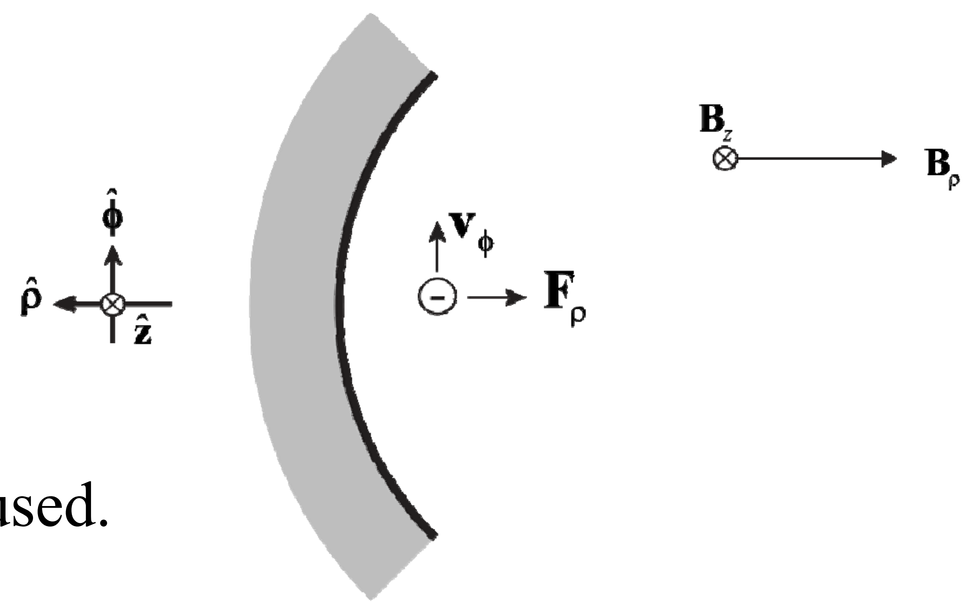


Radial field ⇒ lateral velocity

+

Axial field ⇒ helical motion

Top View



Helical motion ⇒ radial force

⇒Parallel incident beam is focused.

Force on Moving Electron

$\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$ //Lorentz force law

$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$ //electron position

$\mathbf{v} = \dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\varphi}} + \dot{z}\hat{\mathbf{z}}$ //velocity

$\mathbf{B} = B_\rho\hat{\boldsymbol{\rho}} + B_z\hat{\mathbf{z}}$ //magnetic field

$$\mathbf{F} = q\begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ v_\rho & v_\phi & v_z \\ B_\rho & B_\phi & B_z \end{vmatrix} = -e\begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \dot{\rho} & \rho\dot{\phi} & \dot{z} \\ B_\rho & 0 & B_z \end{vmatrix}$$

$$\mathbf{F} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\varphi}} + F_z\hat{\mathbf{z}}$$

$$F_\rho = -eB_z\rho\dot{\phi}$$

$$F_\phi = -eB_\rho\dot{z} + eB_z\dot{\rho}$$

$$F_z = eB_\rho\rho\dot{\phi}$$

Equations of motion

$$\mathbf{F} = m\ddot{\mathbf{r}} = F_\rho \hat{\boldsymbol{\rho}} + F_\phi \hat{\boldsymbol{\varphi}} + F_z \hat{\mathbf{z}}$$

<u>cylindrical↔cartesian</u>

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\dot{\hat{\boldsymbol{\rho}}} = \dot{\phi}\hat{\boldsymbol{\phi}}$$
$$\dot{\hat{\boldsymbol{\phi}}} = -\dot{\phi}\hat{\boldsymbol{\rho}} \qquad //1^{st}\text{ derivatives}$$
$$\dot{\hat{\mathbf{z}}} = \dot{\hat{\mathbf{z}}}$$

$$\ddot{\hat{\boldsymbol{\rho}}} = -\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{\phi}\hat{\boldsymbol{\phi}}$$
$$\ddot{\hat{\boldsymbol{\phi}}} = -\ddot{\phi}\hat{\boldsymbol{\rho}} - \dot{\phi}^2\hat{\boldsymbol{\phi}} \qquad //2^{nd}\text{ derivatives}$$
$$\ddot{\hat{\mathbf{z}}} = \ddot{\hat{\mathbf{z}}}$$

Find force in cylindrical coord's.:

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$
$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$
$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}\right)\hat{\boldsymbol{\varphi}} + \ddot{z}\hat{\mathbf{z}}$$

$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}\right)\hat{\boldsymbol{\varphi}} + \ddot{z}\hat{\mathbf{z}}$$
$$= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\dot{\phi}\right)\hat{\boldsymbol{\varphi}} + \ddot{z}\hat{\mathbf{z}}$$

$$m\ddot{\rho} - m\rho\dot{\phi}^2 = F_\rho = -eB_z\rho\dot{\phi}$$
$$m\cdot\frac{d}{dt}\left(\rho^2\dot{\phi}\right) = \rho F_\phi = -eB_\rho\rho\dot{z} + eB_z\rho\dot{\rho}$$
$$m\ddot{z} = F_z = eB_\rho\rho\dot{\phi}$$

Solve (I): ϕ component

$$m \cdot \frac{d}{dt}\left(\rho^2 \dot{\phi}\right) = -e\rho \dot{z} B_\rho + e\rho\dot{\rho} B_z$$

$$B_\rho = -\frac{\rho B_0}{2} \cdot \left[\delta(z+a) - \delta(z-a)\right]$$ //uniform field in lens

$$B_z = B_0 \cdot [u(z+a) - u(z-a)]$$

$$m \cdot \frac{d}{dt}\left(\rho^2 \dot{\phi}\right) = \frac{e\rho^2 B_0 \dot{z}}{2} \cdot [\delta(z+a) - \delta(z-a)] + e\rho\dot{\rho} B_0 \cdot [u(z+a) - u(z-a)]$$

$$= \frac{d}{dt}\left\{\frac{e\rho^2 B_0}{2} \cdot [u(z+a) - u(z-a)]\right\}$$

Define: $\omega_L \equiv \frac{eB_0}{2m}$ //Larmor Frequency

Assume: $\dot{\phi} = 0 \ \ (z < -a)$

$$\Rightarrow \dot{\phi} = \omega_L \cdot [u(z+a) - u(z-a)] = \begin{cases} 0, & z < -a \\ \omega_L, & -a \le z < a \\ 0, & a \le z \end{cases}$$ //rotation in lens

Solve (II): *z* component

$$m\ddot{z} = e\rho B_\rho \dot{\phi}$$

$$B_\rho = -\frac{\rho B_0}{2}\cdot\left[\delta(z+a)-\delta(z-a)\right]$$

$$\ddot{z} = -\omega_L \rho^2 \dot{\phi}\cdot\left[\delta(z+a)-\delta(z-a)\right]$$

$$\ddot{z} = -(\omega_L\rho)^2\cdot[u(z+a)-u(z-a)]\cdot\left[\delta(z+a)-\delta(z-a)\right]$$

$$\int_{t=0^-}^{0^+}\ddot{z}\cdot\dot{z}\cdot dt = -\omega_L{}^2\cdot\int_{t=0^-}^{0^+}\rho^2[u(z+a)-u(z-a)]\cdot\left[\delta(z+a)-\delta(z-a)\right]\cdot\dot{z}\cdot dt \quad \text{//integrate}$$

Notice: $\frac{d}{dt}\left[u^2(z)\right] = 2u(z)\cdot\delta(z)\cdot\dot{z}$ $\qquad u^2(z) = u(z)$ $\qquad \rho|_{0^-} = \rho|_{0^+}$

Define: $v_z' \equiv \sqrt{v_z{}^2 - (\rho_0\omega_L)^2}$

$$\dot{z} = v_z + (v_z' - v_z)\cdot[u(z+a)-u(z-a)] = \begin{cases} v_z, & z < -a \\ v_z', & -a < z < a \\ v_z, & a < z \end{cases}$$

Small reduction in *z* velocity inside lens for off-axis rays

Solve (III): ρ component

$$m\ddot{\rho} = m\rho\dot{\phi}^2 - eB_z\rho\dot{\phi}$$

$$\dot{\phi}^2 = {\omega_L}^2 \cdot [u(z+a) - u(z-a)]$$

$$B_z\dot{\phi} = B_0\omega_L \cdot [u(z+a) - u(z-a)]$$

$$\ddot{\rho} = -{\omega_L}^2 \cdot [u(z+a) - u(z-a)] \cdot \rho$$

$$\rho(t) = \rho_0 \cdot \cos(\omega_L \cdot t) + C \cdot \sin(\omega_L \cdot t)$$ //solution for $-a < z < a$

$$z = v_z' \cdot t - a$$ $z = -a \quad @t = 0$

$$\rho(z) = \rho_0 \cdot \cos[k' \cdot (z+a)] + C \cdot \sin[k' \cdot (z+a)]$$ //Focusing!

$$k' \equiv \frac{\omega_L}{v_z'} = \frac{\omega_L}{\sqrt{{v_z}^2 - (\rho_0\omega_L)^2}}$$

$$\approx \frac{\omega_L}{v_z} + \frac{{\rho_0}^2{\omega_L}^3}{2{v_z}^3}$$

$$= k + \Delta k$$

$$k \equiv \frac{\omega_L}{v_z}$$

$$\Delta k \equiv \frac{1}{2}{\rho_0}^2 k^3$$

The lens exhibits spherical aberration.

Find paraxial ray

Assume: $k' = k$

We have: $\rho(z) = \rho_0 \cdot \cos[k \cdot (z+a)] + (\tan\theta/k) \cdot \sin[k \cdot (z+a)]$

Paraxial Ray: $\left.\frac{d\rho}{dz}\right|_{z=-a} = 0$ $\Rightarrow \rho(z) = \rho_0 \cdot \cos[k \cdot (z+a)]$

Find trajectory exiting lens:

$\rho(z)\big|_{z=a} = \rho_0 \cdot \cos(2ka)$ $\left.\frac{d\rho}{dz}\right|_{z=a} = -\rho_0 \cdot \sin(2ka)$

$\rho_+(z) = \rho_0 \cdot [-k \cdot \sin(2ka) \cdot (z-a) + \cos(2ka)]$ //ray for $z>a$

Find focal length

Find back focal point:

$$\rho_+(z)\Big|_{z=z_f} = 0 \qquad \frac{z_f}{a} = 1 + \frac{1}{ka\cdot\tan(2ka)}$$

Find back principal plane:

$$\rho_+(z)\Big|_{z=z_0} = \rho_0 \qquad \frac{z_0}{a} = 1 - \frac{\tan(ka)}{ka}$$

$$\frac{f}{a} = \frac{z_f}{a} - \frac{z_0}{a} = \frac{\tan(ka)}{ka} + \frac{1}{ka\cdot\tan(2ka)} = \frac{1}{ka\cdot\sin(2ka)}$$ //Focal length

Find Ray II: Through Lens Center

$$\rho(z) = \rho_0 \cdot \cos[k \cdot (z+a)] + C \cdot \sin[k \cdot (z+a)]$$

$$\rho(z)\big|_{z=0} = 0 \qquad \Rightarrow C = -\rho_0 / \tan(ka)$$

$$\rho(z) = \rho_0 \cdot \frac{\sin(kz)}{\sin(ka)} \qquad \left.\frac{d\rho}{dz}\right|_{z=a} = \frac{k\rho_0}{\tan(ka)}$$

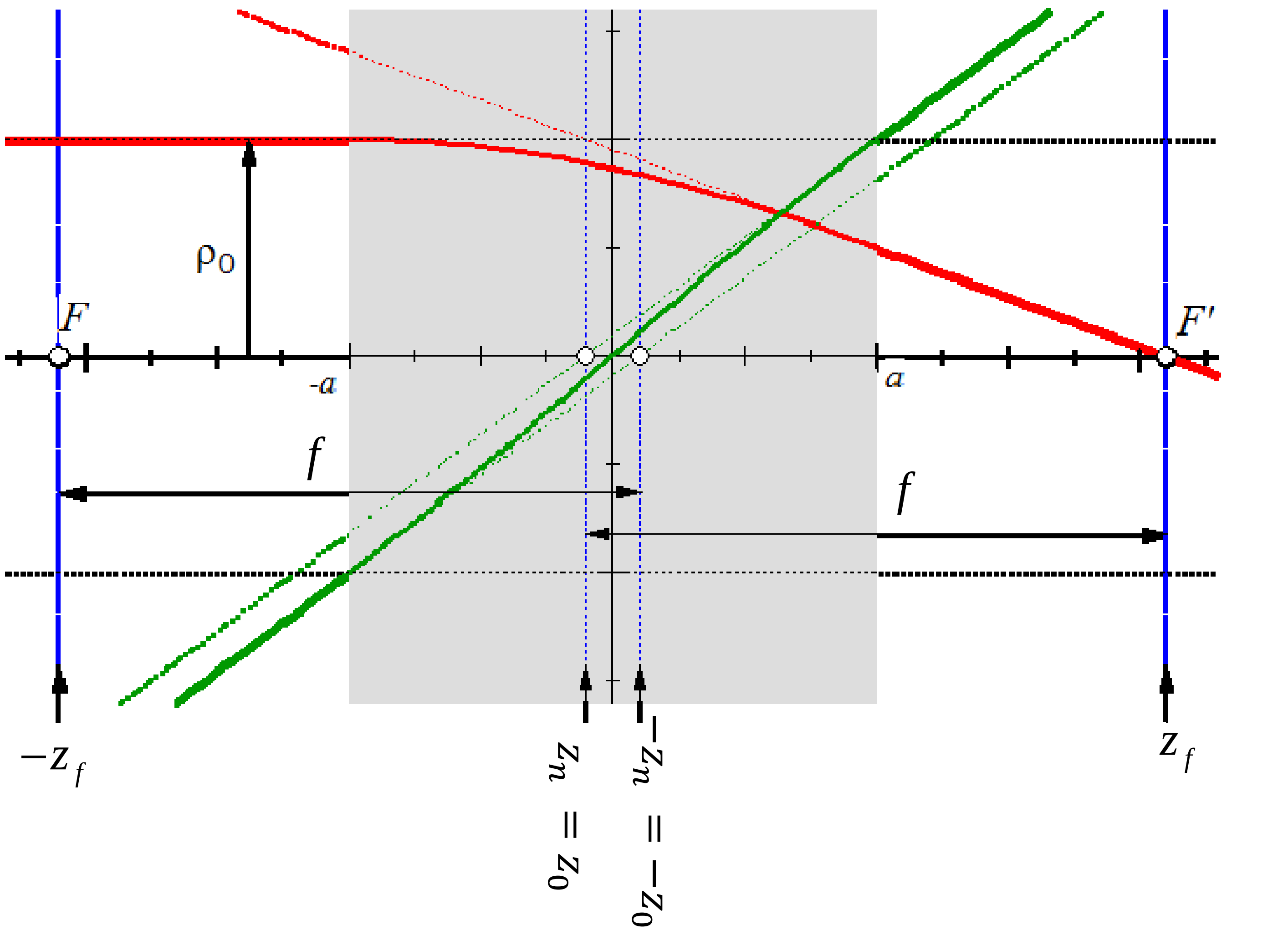
$$\rho_+(z) = \rho_0 \cdot \left[1 + \frac{k}{\tan(ka)} \cdot (z-a)\right] \qquad \text{//ray for } z>a$$

Find back nodal plane:

$$\rho_+(z)\big|_{z=z_n} = 0 \qquad \frac{z_n}{a} = 1 - \frac{\tan(ka)}{ka} = \frac{z_0}{a}$$

Nodal and principal planes coincide.

Ray Diagram: Uniform **B** Lens



Rays follow straight-line paths outside of lens.

Electron trajectory: : Uniform **B** Lens

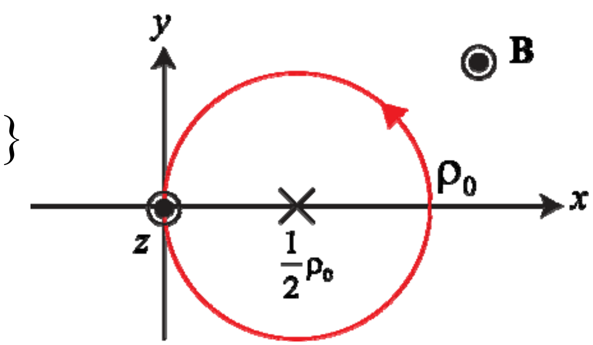
Eqn's of motion: $$\frac{d^2\rho}{dz^2} = -k^2\rho$$ $$\frac{d\phi}{dz} = k$$

Paraxial ray: $$\phi(z) = k \cdot (z + a)$$ $$\rho(z) = \rho_0 \cdot \cos[\phi(z)]$$

Change to Cartesian coord's:

$$x(z) = \rho(z) \cdot \cos[\phi(z)] = \frac{1}{2}\rho_0 \cdot \{1 + \cos[2\phi(z)]\}$$

$$y(z) = \rho(z) \cdot \sin[\phi(z)] = \frac{1}{2}\rho_0 \cdot \sin[2\phi(z)]$$

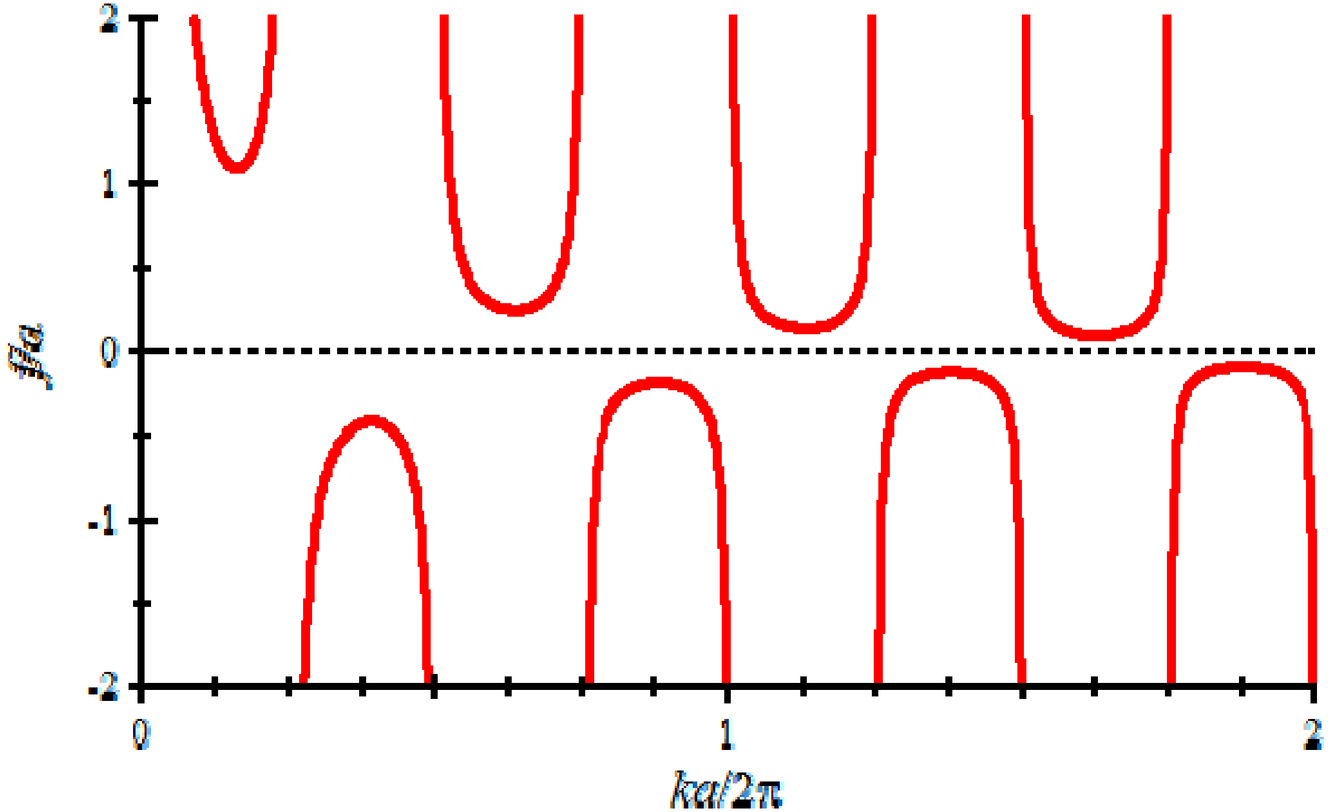


$$2\phi(z)\big|_{z=\lambda-a} = 2\pi \rightarrow \phi(z)\big|_{z=\lambda-a} = \pi \rightarrow k\lambda = \pi \rightarrow \lambda = \frac{\pi}{k} = \frac{v_z \cdot \pi}{\omega_L}$$

$$f = \frac{v_z}{\lambda} = \frac{\omega_L}{\pi} = \frac{\omega}{2\pi} = \frac{1}{T} \rightarrow T = \frac{T_L}{2}$$

Electrons move in circular orbit, but not about the lens center.

Focal length



Increasing field →

Estimate Spherical Aberration Coefficient

$$f \approx f\big|_{k=k_0} + \frac{\partial f}{\partial k}\bigg|_{k=k_0} \cdot \Delta k = f_0 - \Delta f$$ //Expand in Taylor's series

$$k_0 = \frac{\omega_L}{v_z}$$

$$f\big|_{k=k_0} = \frac{1}{k_0 \cdot \sin(2k_0 a)}$$

$$\frac{\partial f}{\partial k}\bigg|_{k=k_0} = -f_0 \cdot \left[\frac{1}{k_0} + \frac{2a}{\tan(2k_0 a)}\right]$$

$$\Delta k = \frac{1}{2}\rho_0^{\ 2} k_0^{\ 3}$$

$\sin(2k_0 a) \to 1$ //Assume strong excitation

$$f\big|_{k=k_0} \approx \frac{1}{k_0} \qquad \frac{\partial f}{\partial k}\bigg|_{k=k_0} \approx -\frac{f_0}{k_0}$$

$$\Delta f = \frac{\rho_0^{\ 2}}{2 f_0}$$

$$f_0 = \frac{1}{k_0} \qquad \Delta f = \frac{f_0}{k_0} \cdot \Delta k$$

$$\theta \approx \frac{\rho_0}{f_0} \qquad \Delta f \approx \frac{1}{2} \cdot f_0 \cdot \theta^2 = C_s \cdot \theta^2 \qquad C_s \approx \frac{f_0}{2} = \frac{v_z}{2\omega_L} = \frac{m v_z}{e B_0}$$