Linear Systems

Describes the output of a linear system

$$G(x) = \int_{x'} F(x') \cdot H(x - x') \cdot dx' = F(x) * H(x)$$

output *input*

Impulse response function

$$H(x) = \int_{x'} \delta(x') \cdot H(x - x') \cdot dx'$$

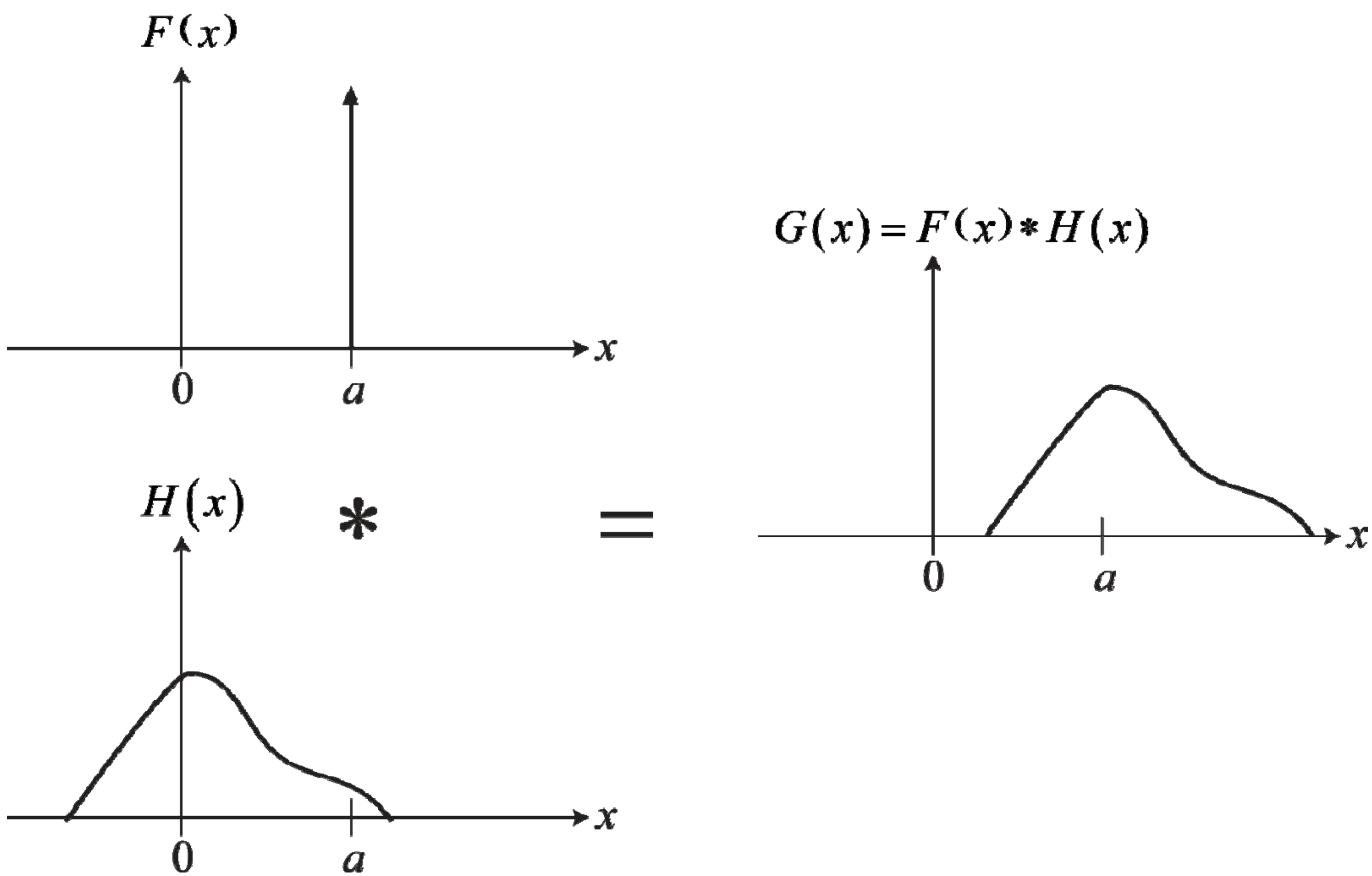
If the microscope is a linear system:

$F(x)$: object

$G(x)$: image

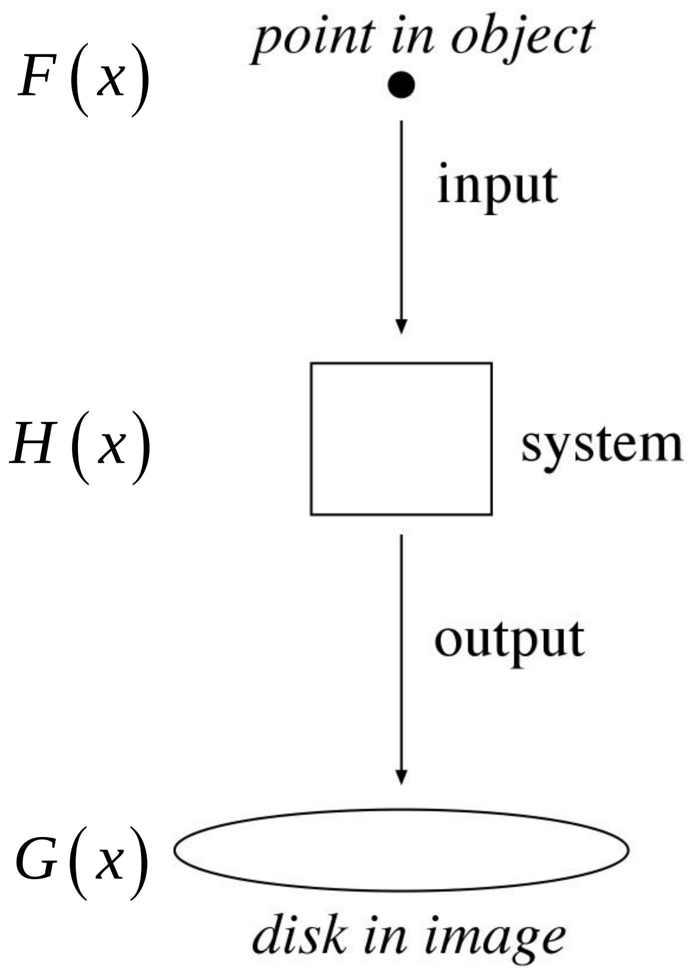
$$G(x) = S\left[F(x)\right]$$

Transfer Function



$H(x)$ is also called the "transfer function" of the system

Microscope as a Linear System



Convolution in direct space:

$$G(x) = F(x) * H(x)$$

Fourier Transforms:

$$F(u) = \Im\{F(x)\}$$

$$G(u) = \Im\{G(x)\}$$

$$H(u) = \Im\{H(x)\}$$

Convolution Theorem:

→Multiplication in reciprocal space

$$G(u) = F(u) \cdot H(u)$$

Contributions to $H(u)$

$$H(u) = A(u) \cdot E(u) \cdot \mathrm{e}^{-i\chi(u)}$$

$A(u)$: aperture function

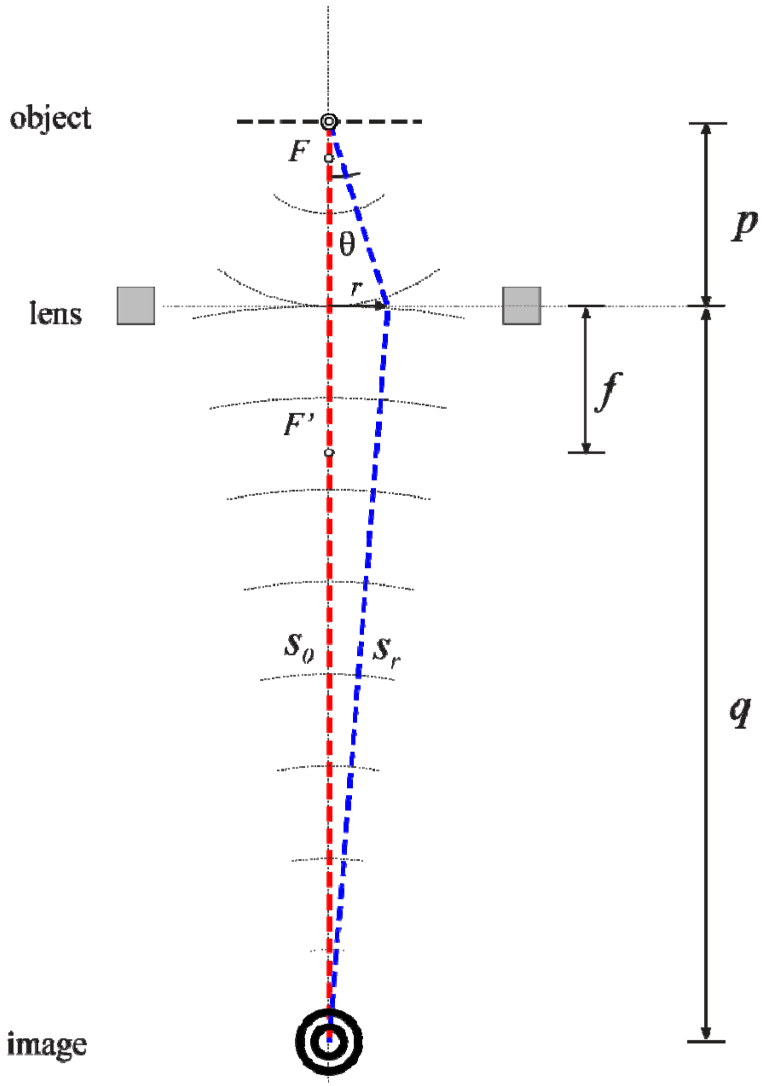
$E(u)$: envelope (damping) function

$\chi(u)$: aberration (phase) function

Most important for phase-contrast imaging:

We need to find $\chi(u)$:

Path Length Correction due to Lens



Ideal lens: $$\frac{1}{p}+\frac{1}{q}=\frac{1}{f_0}$$

$$s_r=\sqrt{p^2+r^2}+\sqrt{q^2+r^2}+\Delta s(r)$$

$$\approx\left(p+\frac{r^2}{2p}\right)+\left(q+\frac{r^2}{2q}\right)+\Delta s(r)$$

$$=p+q+\frac{r^2}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\Delta s(r)$$

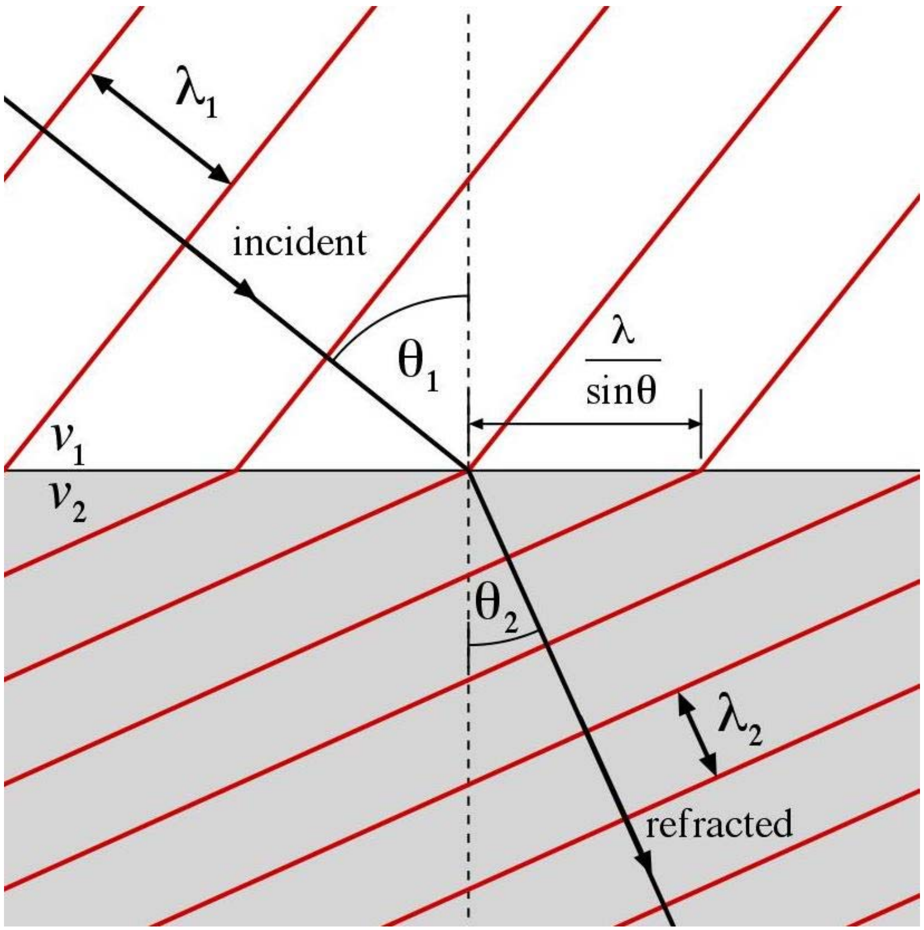
$$s_r=p+q+\frac{r^2}{2f_0}+\Delta s(r)$$

We expect the same net path length for all focused rays:

$$s_r=s_0=p+q \qquad \Rightarrow \Delta s(r)=-\frac{r^2}{2f_0}$$

Snell's Law

The wave crests must be continuous across the interface:



$$\frac{\sin\theta_1}{\lambda_1} = \frac{\sin\theta_2}{\lambda_2}$$

$$\frac{c}{f}\cdot\left(\frac{\sin\theta_1}{\lambda_1} = \frac{\sin\theta_2}{\lambda_2}\right)$$

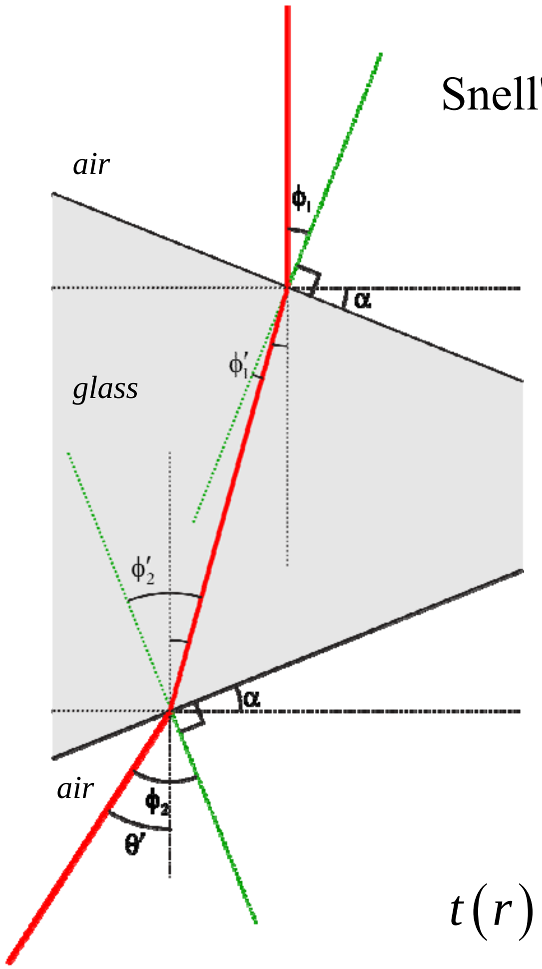
$$v_1 = f\cdot\lambda_1 = c/n_1$$
$$v_2 = f\cdot\lambda_2 = c/n_2$$

refractive indices

$$\Rightarrow \quad n_1\cdot\sin\theta_1 = n_2\cdot\sin\theta_2$$

This applies to any type of wave (not only light).

Thin Optical Lens (I)



Snell's law: $\sin\phi_1 = n \cdot \sin\phi_1'$

$\sin\phi_2 = n \cdot \sin\phi_2'$

$$\phi_1 = \alpha$$

$$\phi_1' + \phi_2' = 2\alpha$$

$$\theta' = \phi_2 - \alpha$$

Small angles: $\phi_1 \approx n\phi_1'$

$\phi_2 \approx n\phi_2'$

$$\theta' = n\phi_2' - \alpha = n\left(2\alpha - \phi_1'\right) - \alpha = 2(n-1)\alpha$$

$$\tan\theta' = \frac{r}{f_0} \approx \theta' \qquad \alpha = \frac{r}{2(n-1)\,f_0}$$

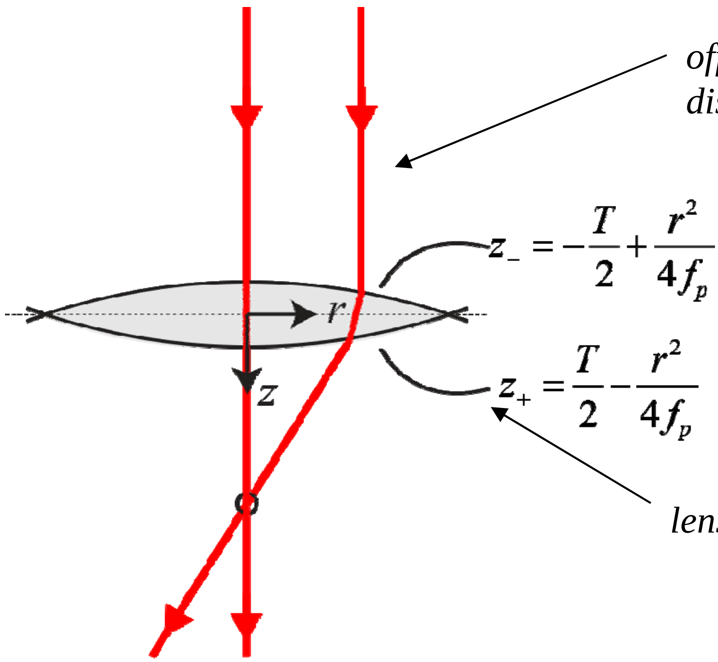
$$\frac{dt}{dr} = -2\tan\alpha \approx -2\alpha = -\frac{r}{(n-1)\,f_0}$$

$$t(r) = T - \int_{r'=0}^{r} \frac{r' \cdot dr'}{(n-1)\,f_0} = T - \frac{r^2}{2(n-1)\,f_0} = T - \Delta t(r)$$

Thickness change: $\Delta t(r) = \dfrac{r^2}{2(n-1)\,f} = \dfrac{r^2}{2\,f_p}$ ← parabola focal length

Optical focal Length: $f_0 = \dfrac{f_p}{n-1}$

Thin Optical Lens (II)



Wavelength:

vacuum: $\lambda_0 = c/f$

medium: $\lambda = \frac{c}{nf} = \lambda_0/n$

Phase difference:

$$\Delta\phi(r) = 2\pi\cdot[T - t(r)]\cdot\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$$= -2\pi\cdot\frac{(n-1)\cdot\Delta t(r)}{\lambda_0}$$

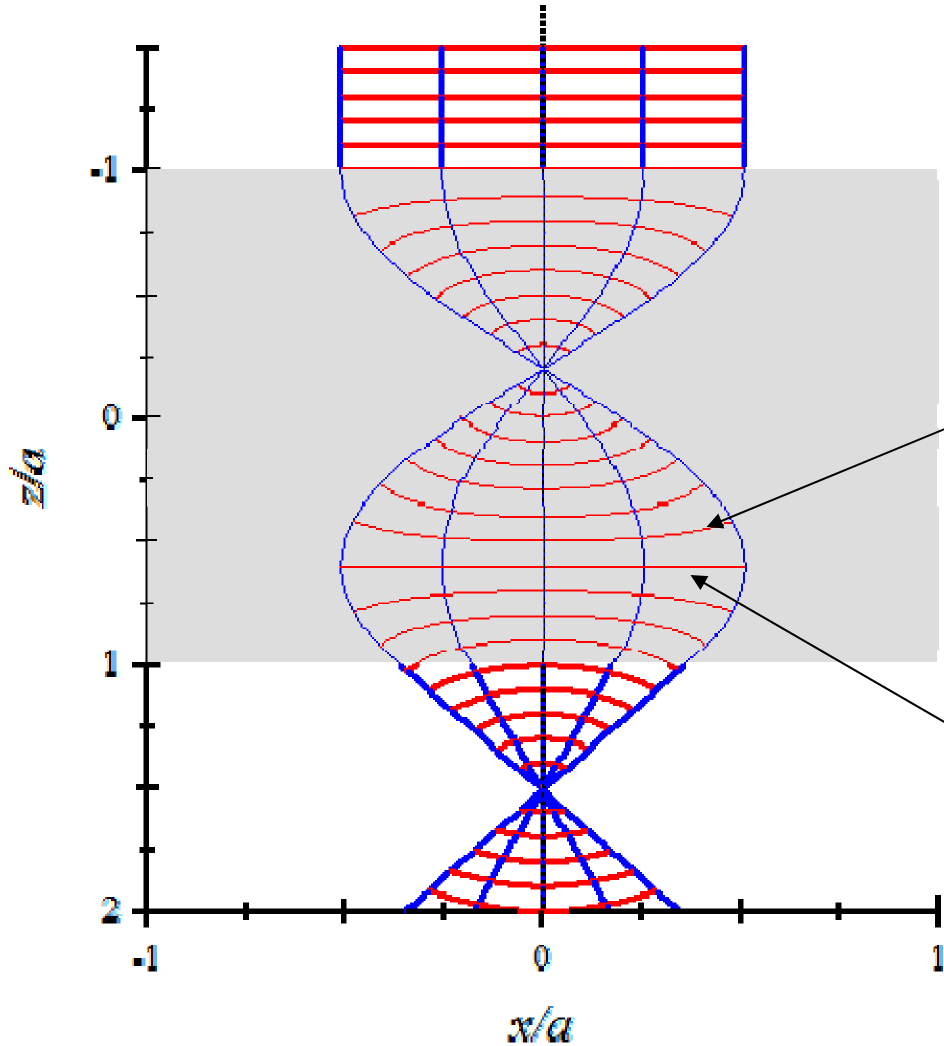
$$\Delta\phi(r) = 2\pi\cdot\left(\frac{\Delta s(r)}{\lambda_0}\right)$$

$$\frac{d(\Delta t)}{dr} = \frac{r}{(n-1)\,f_0} \qquad \Delta s(r) = -(n-1)\cdot\Delta t(r) \qquad \frac{d(\Delta s)}{dr} = -\frac{r}{f_0}$$

Optical path-length difference:

$$\Delta s(r) = -\int_{r'=0}^{r}\frac{r'\cdot dr'}{f_0} = -\frac{r^2}{2\,f_0}$$

Uniform-Field Electron Lens



off-axis rays have a greater distance to travel

less momentum, longer wavelength

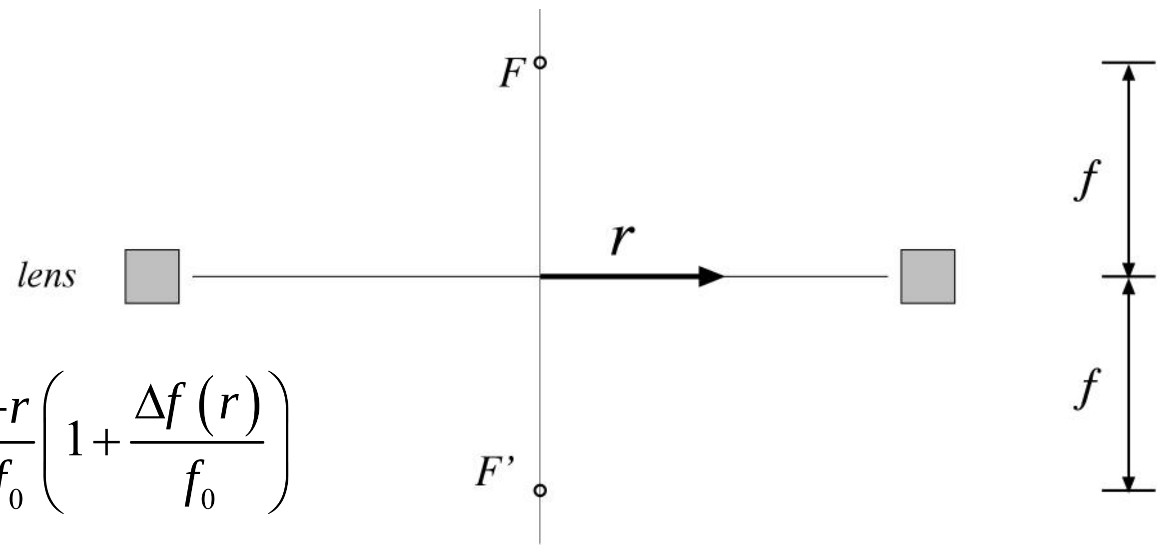
Path Length Correction due to Lens

A path-length correction for each trajectory through the lens should be included.

Generalize: $$\frac{d(\Delta s)}{dr} = \frac{-r}{f_0} \rightarrow \frac{-r}{f(r)}$$

Non-ideal lens:

$$f(r) = f_0 - \Delta f(r)$$



$$\frac{d(\Delta s)}{dr} = \frac{-r}{f_0 - \Delta f(r)} \approx \frac{-r}{f_0}\left(1 + \frac{\Delta f(r)}{f_0}\right)$$

$$\Delta s(r) = \frac{-r^2}{2 f_0} - \int_{r'=0}^{r} \left[\frac{r' \cdot \Delta f(r')}{f_0{}^2}\right] \cdot dr'$$

The derivative of the path-length difference determines the focal length.

Influence of Spherical Aberration

From Chap. 6: $$\Delta f\left(r\right)=C_s\cdot\left(\frac{r}{f_0}\right)^2$$

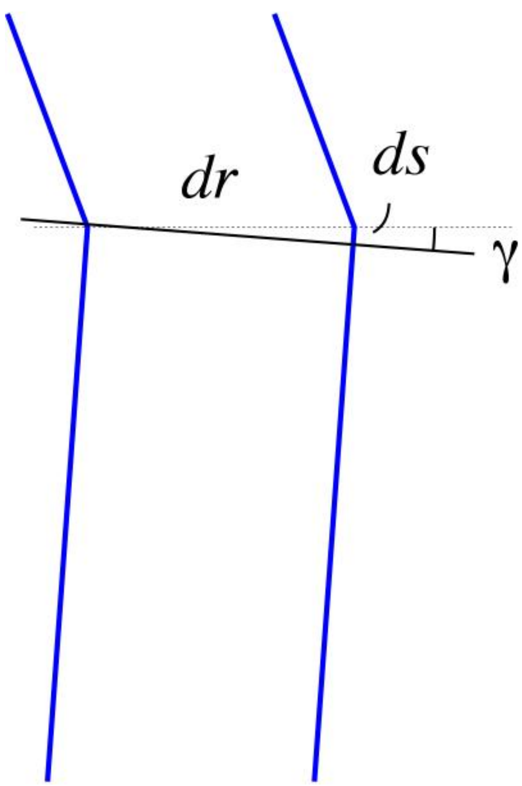
$$\Delta s\left(r\right)=\frac{-r^2}{2f_0}-\int_{r'=0}^{r}\left(\frac{r'}{f_0{}^2}\right)\cdot\left[C_s\cdot\left(\frac{r'}{f_0}\right)^2\right]dr'$$

$$=\underbrace{\frac{-r^2}{2f_0}}_{\text{normal focusing}}-\underbrace{\frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4}_{\text{aberration}}$$

Spherical aberration (wave optics)

When object is at focal point, if no aberration,
all rays on image side (back) of lens will be parallel to lens

With aberration, off-axis rays will be tend toward the optic axis

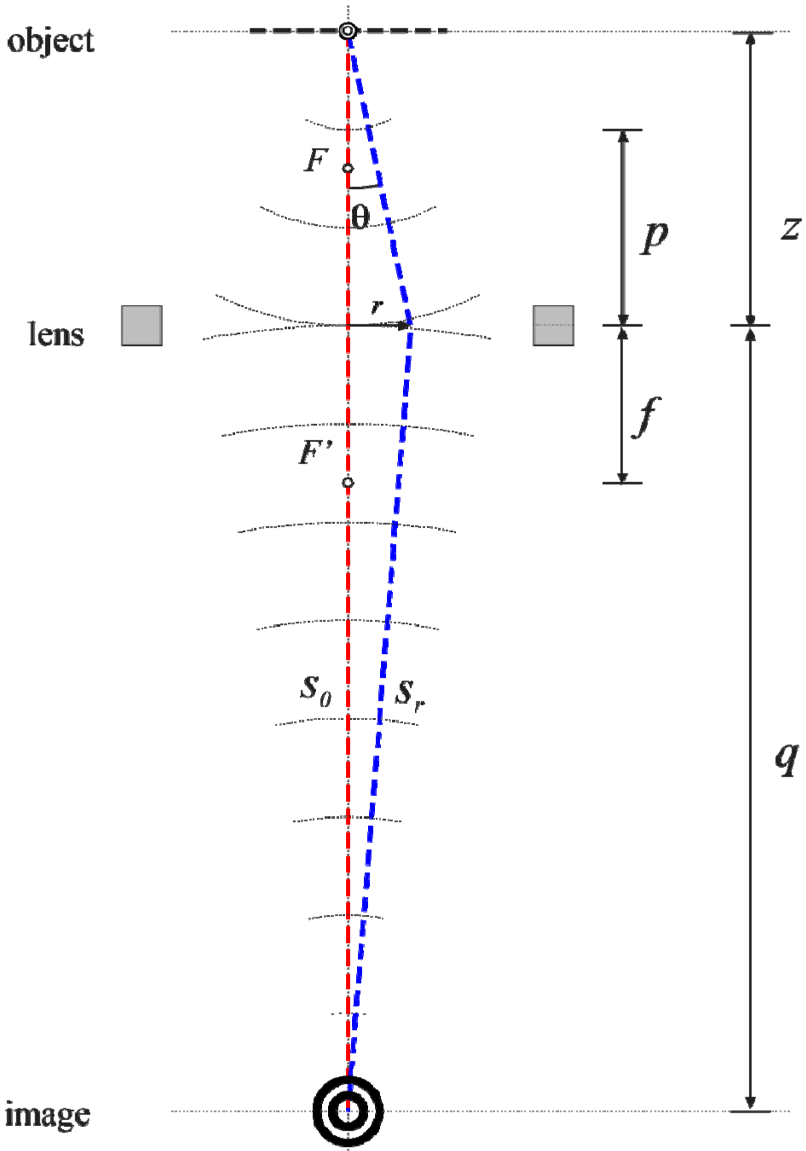


$$\gamma = \frac{C_s r^3}{f_0^{\,4}} \qquad \gamma = \frac{ds}{dr}$$

$$ds = \gamma \cdot dr = \frac{C_s r^3}{f_0^{\,4}} \cdot dr$$

$$\Delta s(r) = \frac{C_s}{f_0^{\,4}} \int_{r'=0}^{r} r'^3 \cdot dr' = \frac{C_s r^4}{4 f_0^{\,4}}$$

Path-length difference: general case



Assume object is on-axis, but not in focus.

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f_0}$$

Axial Ray: $s_0 = z + q$

Lens: $\Delta s(r) = \frac{-r^2}{2f} - \frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4$

Off-axis Ray:

$$s_r = \sqrt{z^2+r^2} + \sqrt{q^2+r^2} + \Delta s(r)$$

$$\approx z + \frac{r^2}{2z} + q + \frac{r^2}{2q} + \Delta s(r)$$

$$s_r = s_0 + \Delta s_{net}(r)$$

$$\Delta s_{net}(r) = \frac{r^2}{2}\cdot\left(\frac{1}{z}+\frac{1}{q}-\frac{1}{f}\right) - \frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4$$

Combine defocus and sample height

$$p = z + \Delta z$$

$$f = f_0 - \Delta f$$

$$\frac{1}{z} + \frac{1}{q} - \frac{1}{f} = \frac{1}{p - \Delta z} + \frac{1}{q} - \frac{1}{f_0 - \Delta f}$$

$$\approx \cancel{\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{f_0} \right)} + \frac{\Delta z}{p^2} - \frac{\Delta f}{f_0^{\,2}}$$

$$\approx \frac{\Delta z}{p^2} - \frac{\Delta f}{f_0^{\,2}}$$

$$M_0 = \frac{q}{p} = \frac{q}{f_0} - 1 \approx \frac{q}{f_0} \gg 1 \rightarrow f_0 \ll q$$

$$\frac{1}{z} + \frac{1}{q} - \frac{1}{f} \approx \frac{\Delta z - \Delta f}{f_0^{\,2}}$$

$$p \approx f_0$$

only the difference matters

Optical path-length difference

$$\Delta s_{net} \approx \frac{1}{2}\cdot\left(\Delta z - \Delta f\right)\cdot\left(\frac{r}{f_0}\right)^2 - \frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4$$

$$r = z\cdot\theta$$

$$\Delta s = \frac{1}{2}\cdot\left(\Delta z - \Delta f\right)\cdot\left(\frac{z}{f_0}\right)^2\cdot\theta^2 - \frac{1}{4}C_s\left(\frac{z}{f_0}\right)^4\cdot\theta^4$$

$$\frac{z + \cancel{\Delta z}}{f_0} = \frac{1}{1 - f_0/q} \approx 1 + \cancel{f_0/q} \rightarrow z \approx f_0$$

Net Effect: $$\Delta s \approx \frac{1}{2}\cdot\left(\Delta z - \Delta f\right)\cdot\theta^2 - \frac{1}{4}C_s\cdot\theta^4$$

Phase correction

Phase shift: $$\Delta\phi = -2\pi\left(\frac{\Delta s}{\lambda}\right) = \frac{\pi}{\lambda}\cdot\left[\left(\Delta f - \Delta z\right)\cdot\theta^2 + \frac{1}{2}C_s\theta^4\right]$$

Scattering angle: $$\theta = \frac{R}{L} = \frac{\lambda}{d} = \lambda u$$

Frequency representation: $$\chi\left(u\right) = \pi\cdot\left(\Delta f - \Delta z\right)\cdot\lambda u^2 + \frac{\pi}{2}C_s\lambda^3 u^4$$

Image function for weak-phase object

A weak phase object produces an image function:

$$G\left(x\right)=F\left(x\right)*H\left(x\right)$$

$$G\left(x\right)=\left[1+i\sigma V_t\left(x\right)\right]*H\left(x\right)$$

First term: $\Im\left\{1*H\left(x\right)\right\}=\Delta\left(u\right)\cdot H\left(u\right)$

$$\Im^{-1}\left\{\Delta\left(u\right)\cdot H\left(u\right)\right\}=H\left(u\right)\Big|_{u=0}$$

The overall phase doesn't matter, so we might as well pick

$$H\left(u\right)\Big|_{u=0}=1$$

$$G\left(x\right)\approx 1+i\sigma V_t\left(x\right)*H\left(x\right)$$

Simple situation

We might expect the ideal transfer function to be:

$$H(x) = \delta(x)$$

$$G(x) = \left[1 + i\sigma V_t(x)\right] * H(x) = 1 + i\sigma V_t(x)$$

Intensity:

$$I(x) = \left|G(x)\right|^2 = 1 + \cancel{\left[\sigma V_t(x)\right]^2} \approx 1$$

This gives no contrast to first order in σ

We will need a relative phase shift between the incident and scattered beams to see phase contrast

Contrast transfer function

Intensity:

$$
\begin{aligned}
I(x) &= \left|G(x)\right|^2 \\
&= \left[1 - i\sigma V_t(x) * H^*(x)\right] \cdot \left[1 + i\sigma V_t(x) * H(x)\right] \\
&\approx 1 + i\sigma V_t(x) * \left[H(x) - H^*(x)\right] \\
&= 1 - 2\sigma V_t(x) * \mathrm{Im}\left[H(x)\right]
\end{aligned}
$$

$$
I(x) = 1 - \sigma V_t(x) * T(x)
$$

$$
T(x) \doteq 2\,\mathrm{Im}\left[H(x)\right]
$$

CTF in reciprocal space

$$I(x) = 1 - \sigma V_t(x) * T(x)$$

$$I(u) = \Im\left[1 - \sigma V_t(x) * T(x)\right] = \Delta(u) - \sigma V_t(u) \cdot T(u)$$

$$\begin{aligned}
T(u) &= \Im[T(x)] \\
&= \Im\left\{2 \cdot \mathrm{Im}\left[H(x)\right]\right\} \\
&= -i \cdot \Im\left[H(x) - H^*(x)\right] \\
&= -i \cdot \int_x dx \cdot \left\{\int_{u'} du' \cdot \left[H(u') \cdot \mathrm{e}^{2\pi i u' x} - H^*(u') \cdot \mathrm{e}^{-2\pi i u' x}\right]\right\} \cdot \mathrm{e}^{-2\pi i u x} \\
&= -i \cdot \int_{u'} du' \cdot \left\{H(u') \cdot \left[\int_x dx \cdot \mathrm{e}^{2\pi i (u'-u)x}\right] - H^*(u') \cdot \left[\int_x dx \cdot \mathrm{e}^{-2\pi i (u'+u)x}\right]\right\} \\
&= -i \cdot \int_{u'} du' \cdot \left\{H(u') \cdot \Delta(u'-u) - H^*(u') \cdot \Delta(u'+u)\right\}
\end{aligned}$$

$$T(u) = -i \cdot \left[H(u) - H^*(-u)\right]$$

Special Case: $H(u)$ is an even function

If $H(-u) = H(u)$

then $H^*(-u) = H^*(u)$

$$T(u) = -i \cdot \left[H(u) - H^*(-u) \right]$$

$$= -i \left[H(u) - H^*(u) \right]$$

$$T(u) = 2 \cdot \mathrm{Im}\left[H(u) \right]$$

Using the CTF

$$H(u) = A(u) \cdot E(u) \cdot \exp\left[-i\chi(u)\right]$$

Assume: $$H(u) = H(-u)$$

$$T(u) = 2\,\mathrm{Im}\left[H(u)\right] = -2A(u)E(u)\sin\chi(u)$$

$$\begin{aligned} I(u) &= \Im\left[1 - \sigma V_t(x) * T(x)\right] \\ &= \Delta(u) - \sigma V_t(u) \cdot T(u) \end{aligned}$$

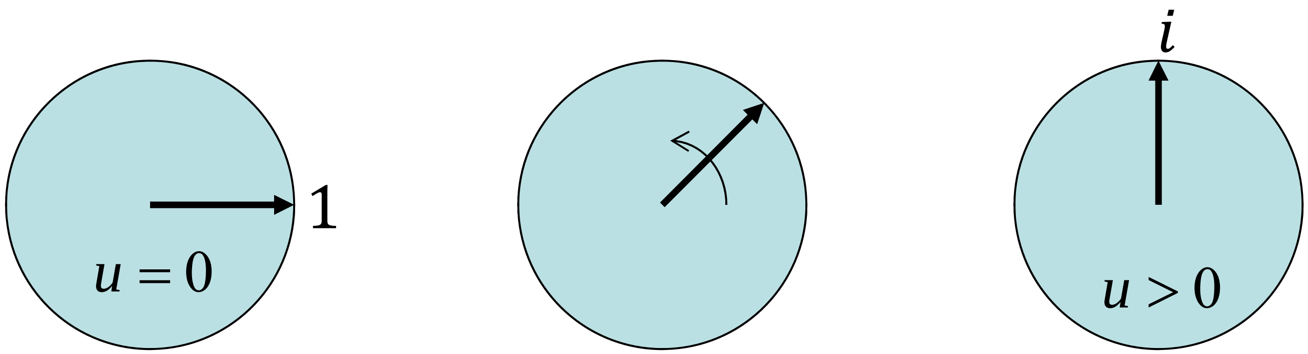
$$I(u) = \Delta(u) + 2\sigma V_t(u) \cdot A(u) \cdot E(u) \cdot \sin\chi(u)$$

Ideal transfer function for phase object (I)

$$\chi(u) = \begin{cases} 0, & |u| < 1/b \\ -\dfrac{\pi}{2}, & |u| \geq 1/b \end{cases}$$

$$H(u) = \mathrm{e}^{-i\chi(u)} = \begin{cases} 1, & |u| < 1/b \\ i, & |u| \geq 1/b \end{cases}$$

$H(u)$

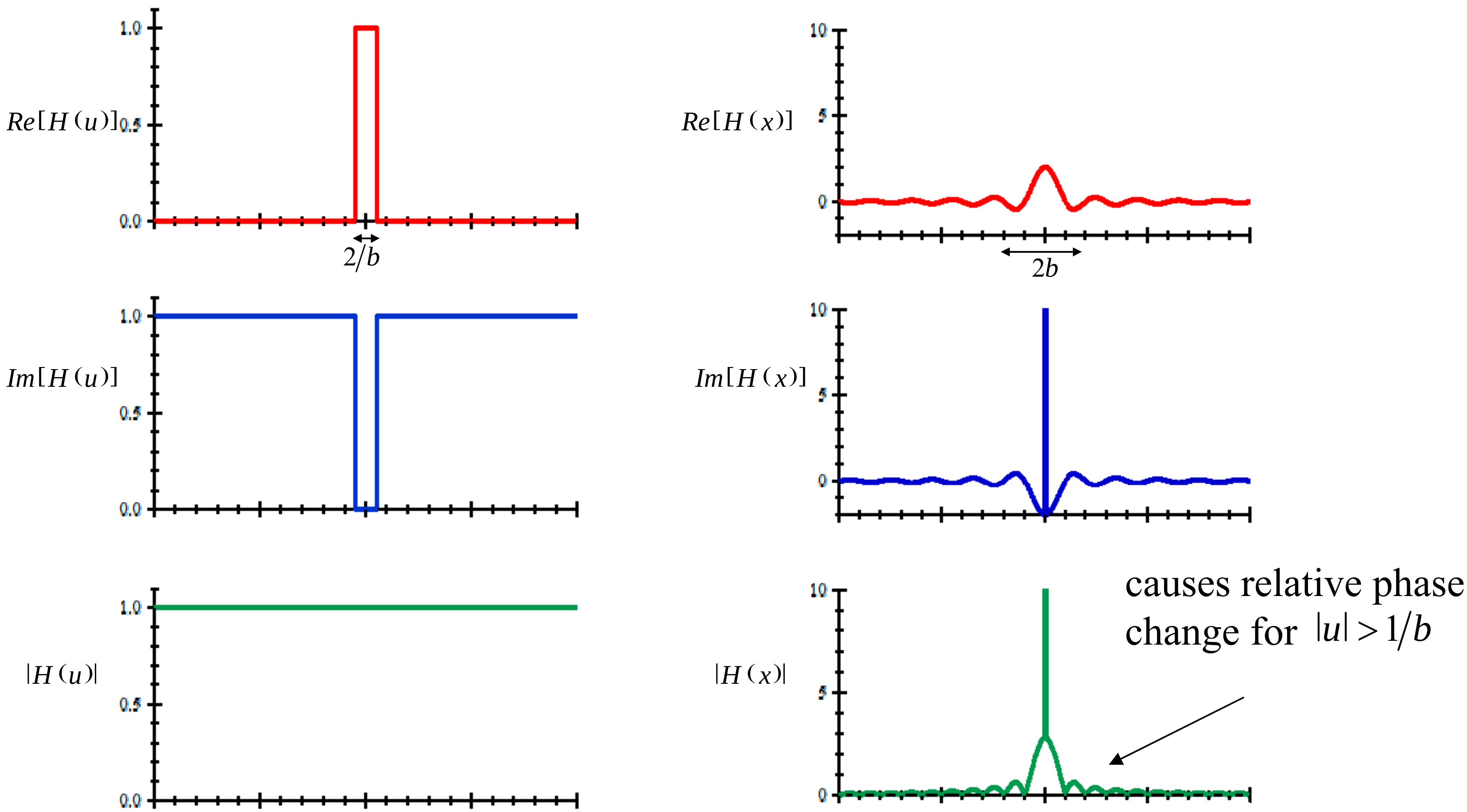


$$T(u) = 2\,\mathrm{Im}\left[H(u)\right] = \begin{cases} 0, & |u| < 1/b \\ 2, & |u| \geq 1/b \end{cases}$$

Ideal transfer function for phase object (II)

$$H(u) = \mathrm{e}^{-i\chi(u)} = \begin{cases} 1, & |u| < 1/b \\ i, & |u| \geq 1/b \end{cases}$$

$$H(x) = i\delta(x) + 2(1-i)\cdot \mathrm{sinc}(2\pi x/b)$$



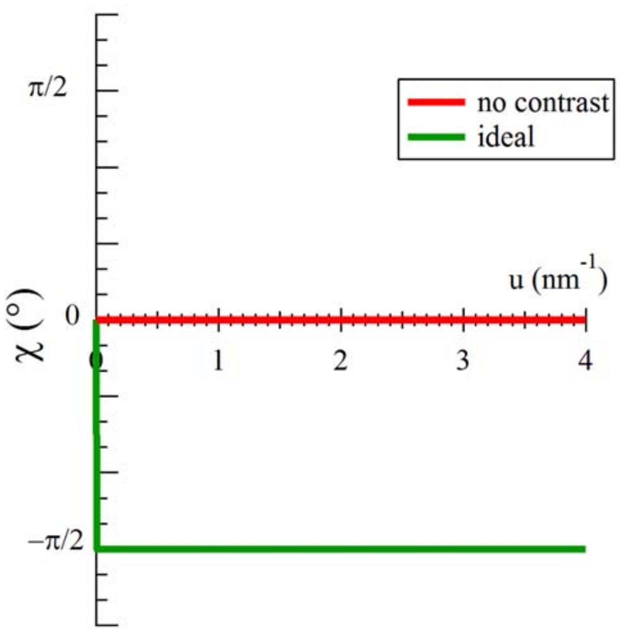
Ideal CTF

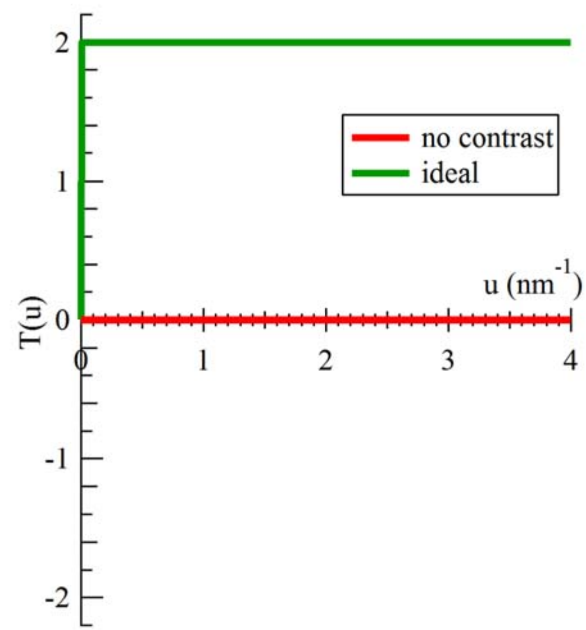
Assume $A(u)E(u)=1$

$$T(u) = -2\sin\chi(u)$$

$$\chi(u) = \begin{cases} 0, & u = 0 \\ -\pi/2, & u > 0 \end{cases}$$

$$T(u) = \begin{cases} 0, & u = 0 \\ 2, & u > 0 \end{cases}$$

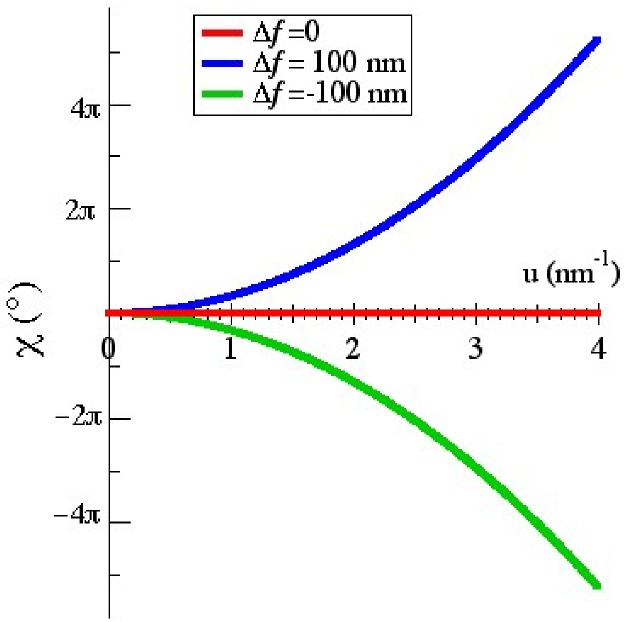


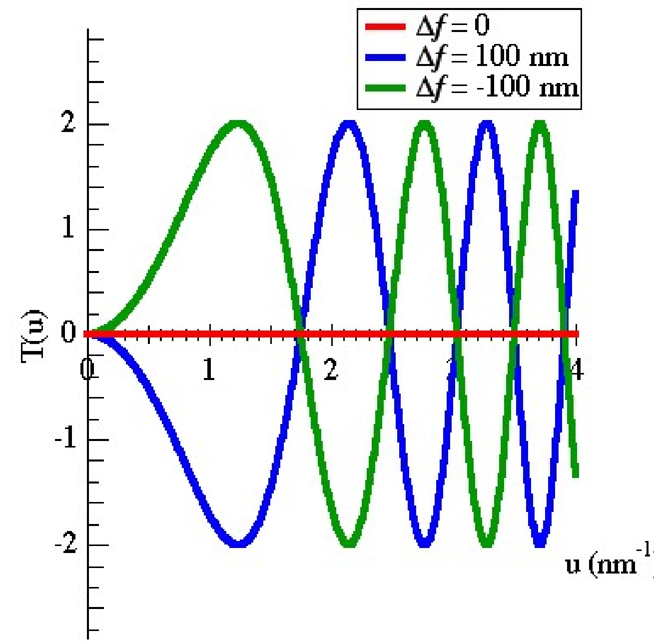


$T(u)>0 \Rightarrow$ positive phase contrast

Real CTF: (case 1: C_s=0)

$$\chi(u) = \pi \cdot \Delta f \cdot \lambda u^2$$

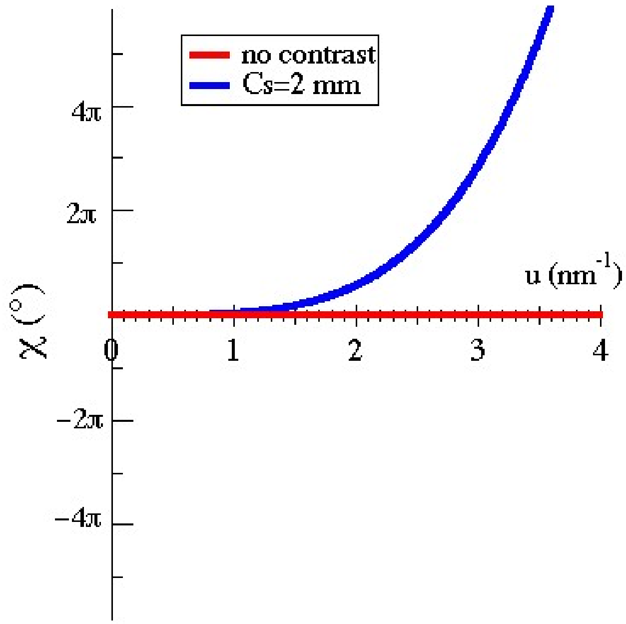


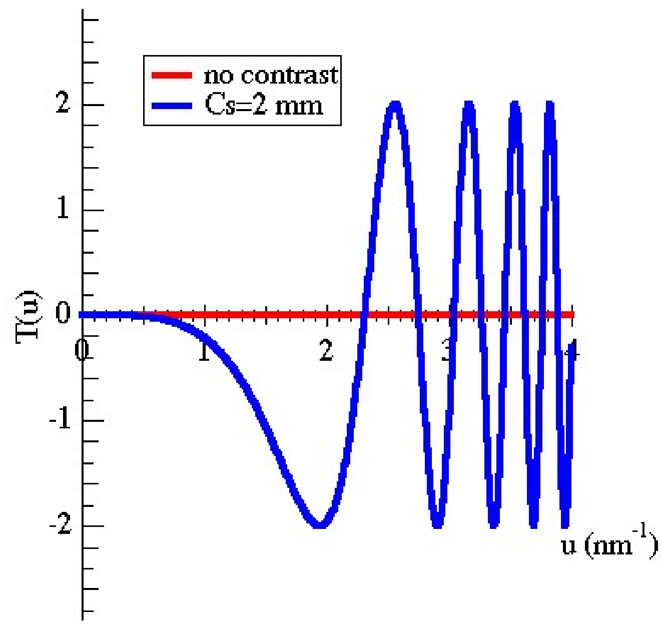


$\Delta f < 0$: underfocusing
$\Delta f > 0$: overfocusing

Real CTF: (case 2: Δf=0)

$$\chi(u) = \frac{1}{2}\pi C_s \lambda^3 u^4$$

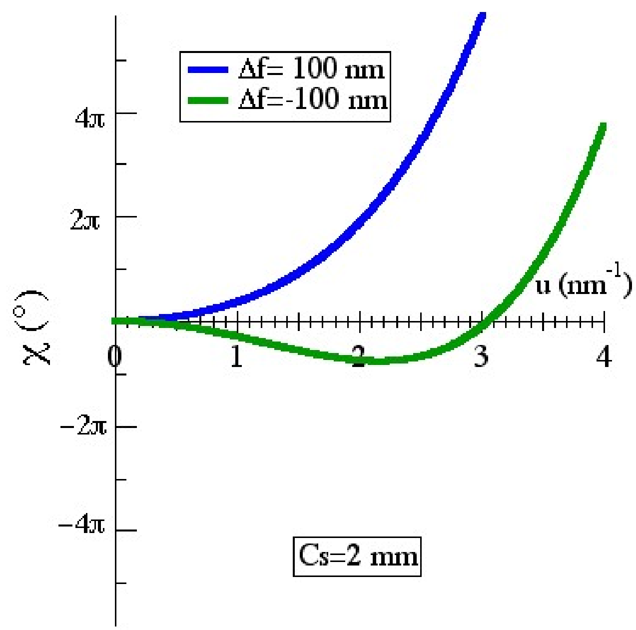


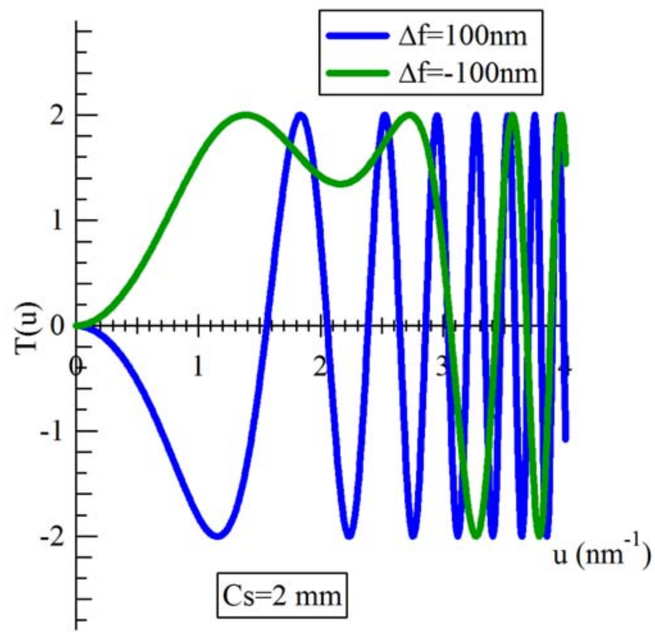


Contributes to negative contrast

Real CTF: (general)

$$\chi(u) = \pi \cdot \Delta f \cdot \lambda u^2 + \frac{1}{2}\pi C_s \lambda^3 u^4$$





Underfocus gives a region with constant phase, positive contrast

Scherzer Defocus

$$\frac{d\chi(u)}{du} = 2\pi \cdot \Delta f \cdot \lambda u + 2\pi C_s \lambda^3 u^3$$

Stationary Phase: $$\left.\frac{d\chi(u)}{du}\right|_{u=u_{\min}} = 0 \Rightarrow \Delta f = \Delta f_{sch} = -C_s \lambda^2 u_{\min}{}^2$$

We need to pick what phase we want stationary:

$$T(u_{\min}) = -2\sin\left[\chi(u_{\min})\right] = \frac{\sqrt{3}}{2} \cdot T_{\max} = \sqrt{3} \qquad \longrightarrow \qquad \sin\left[\chi(u_{\min})\right] = -\frac{\sqrt{3}}{2}$$

$$\chi(u_{\min}) = -\frac{2\pi}{3} = -\frac{1}{2}\pi C_s \lambda^3 u_{\min}{}^4$$

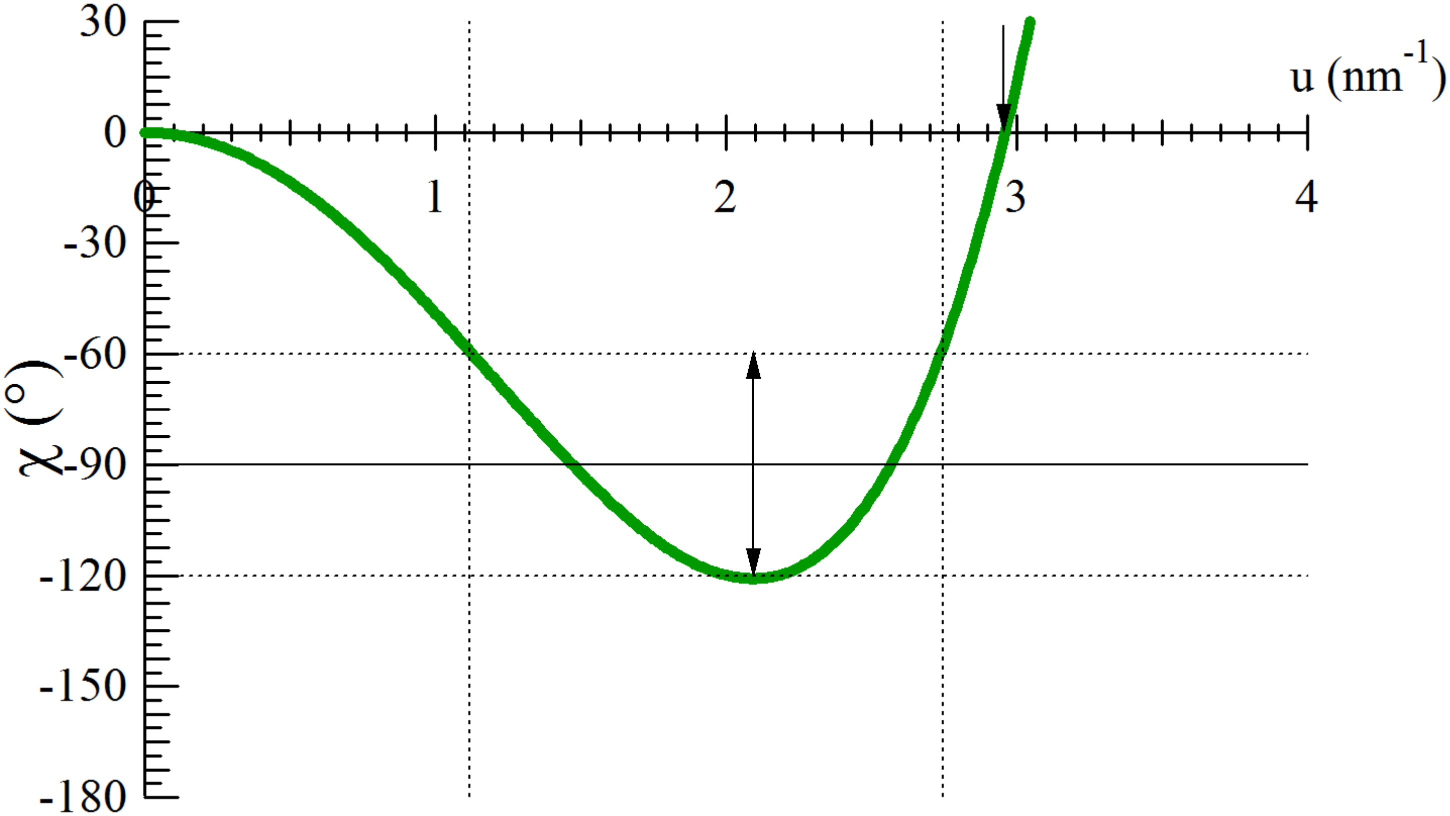
$$u_{\min} = \left(\frac{4}{3C_s\lambda^3}\right)^{1/4} \qquad \Delta f_{sch} = -\left(\frac{4}{3}C_s\lambda\right)^{1/2} \approx -1.2\left(C_s\lambda\right)^{1/2}$$

This choice gives a relatively constant CTF.

Phase At Scherzer defocus

$C_s = 2.0$ mm

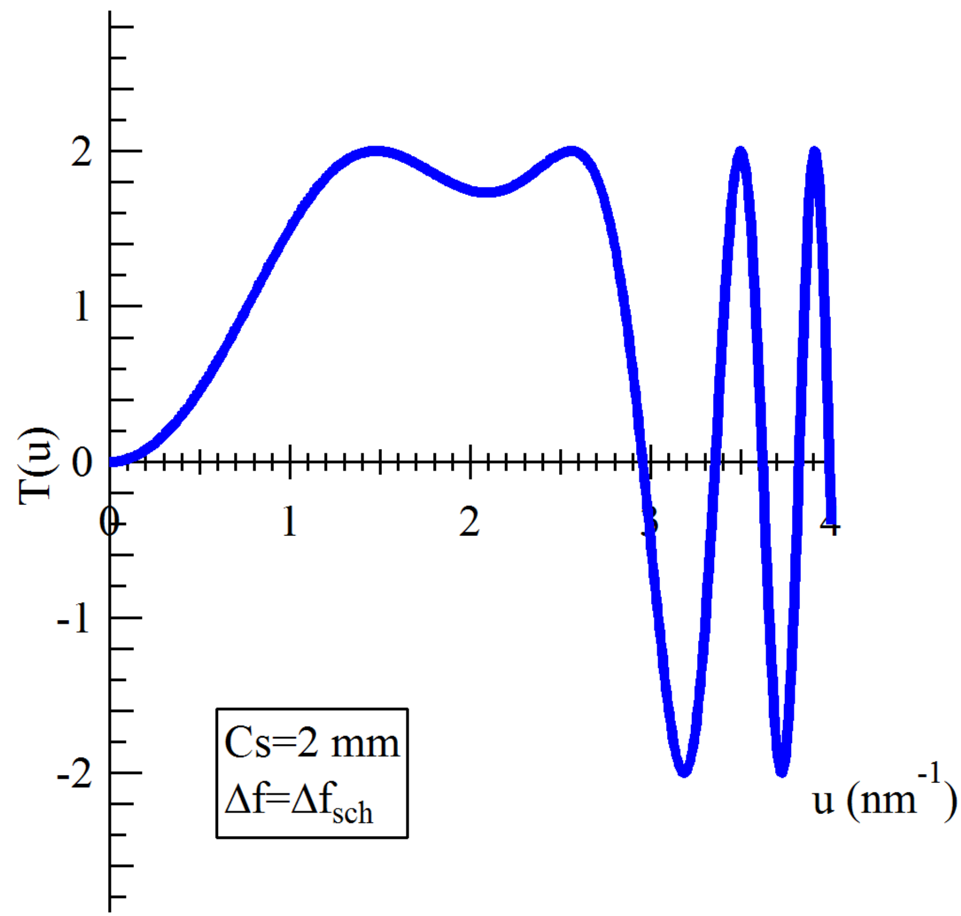
$E = 125$ KeV



CTF at Scherzer defocus

E=125 keV
C_s=2.0 mm

$$\Delta f_{sch} = -93.4\ \text{nm}$$



Resolution at Scherzer defocus

$$\sin\chi\left(u_{sch}\right)=0\Rightarrow\chi\left(u_{sch}\right)=0$$

$$0=\Delta f_{sch}\cdot\lambda u_{sch}^{\ 2}+\frac{1}{2}C_s\lambda^3u_{sch}^{\ 4}$$

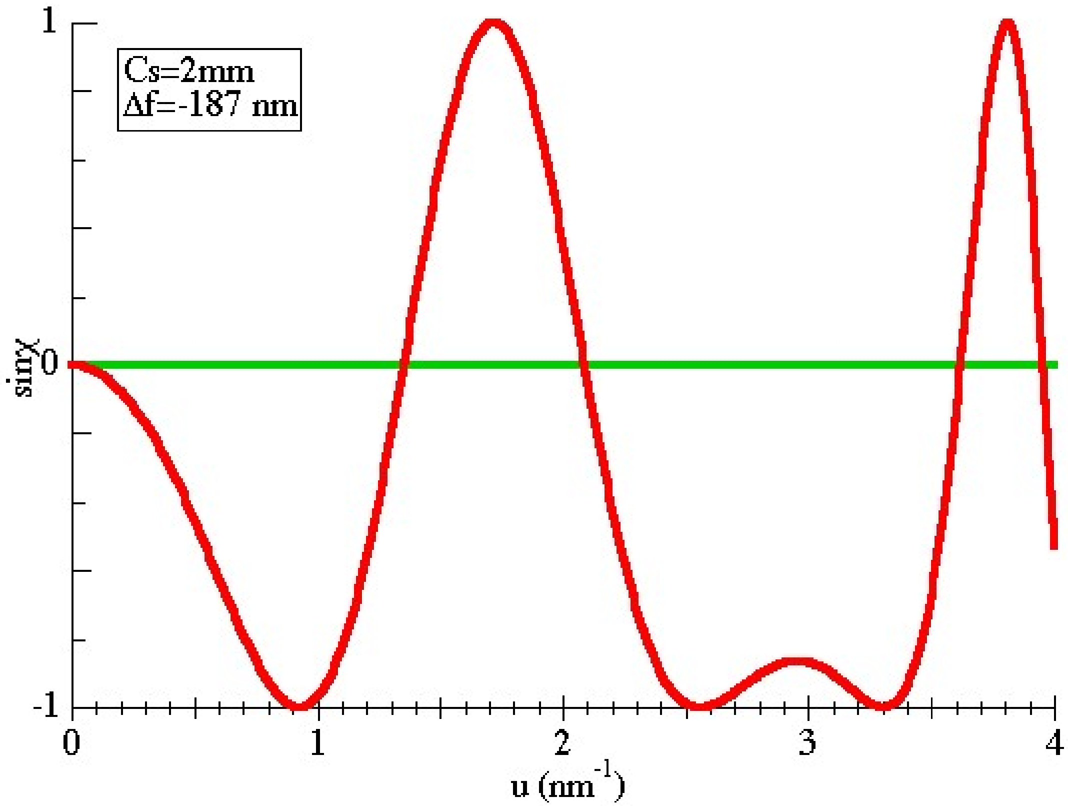
we already know: $$\Delta f_{sch}=-\left(\frac{4}{3}C_s\lambda\right)^{1/2}$$

$$u_{sch}=2\cdot\left(3C_s\lambda^3\right)^{-1/4}\approx 1.51\left(C_s\lambda^3\right)^{-1/4}$$

$$d_{sch}=\frac{1}{u_{sch}}\approx 0.66\left(C_s\lambda^3\right)^{1/4}$$

Passbands

We could make the phase stationary at higher *u*:



Damping due to temporal incoherence

$$E_c(u) = \exp\left(-\pi^2\delta^2\lambda^2u^4/2\right)$$

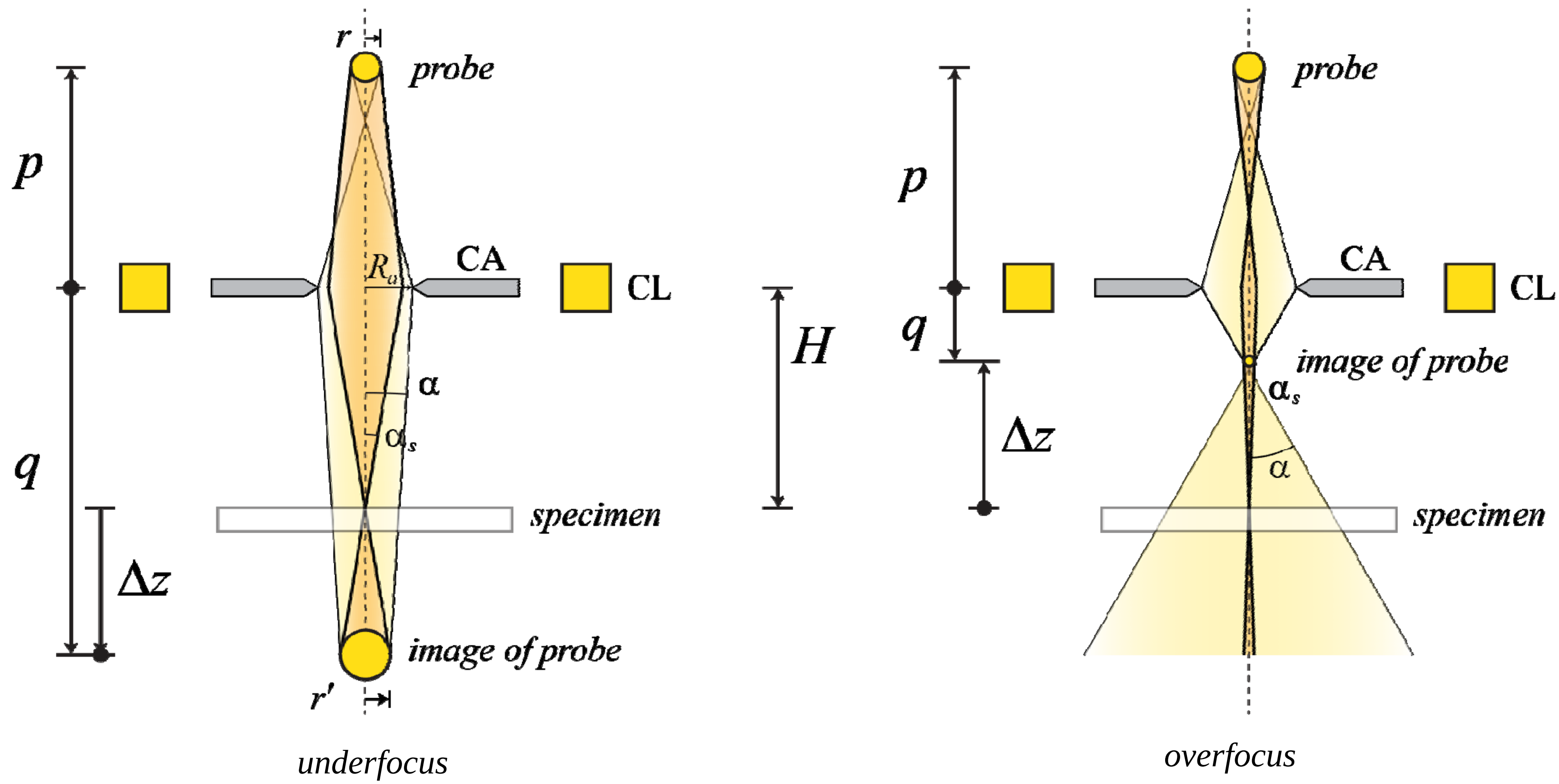
$$\delta = C_c \cdot \sqrt{\left(\frac{\delta E}{E}\right)^2 + \left(\frac{\delta I}{I}\right)^2}$$

chromatic
aberration
coefficient

$\dfrac{\delta E}{E}$: Variation in energy

$\dfrac{\delta I}{I}$: Variation in objective-lens current

Beam convergence: one-lens condenser (I)



A range of illumination angles may be incident on each sample point.
The illumination semiangle α_s is not the same as the beam convergence angle α.

Beam convergence: one-lens condenser (II)

Semi-angle of illumination at a point on the specimen:

$$\alpha_s = \left|\frac{r'}{\Delta z}\right| = \left|\frac{H}{\Delta z} - 1\right| \cdot \alpha_r$$

Illumination semi-angle in lens plane: $\alpha_r = \dfrac{r}{p}$

Max. illumination angle limited by aperture: $\alpha_s \le \alpha_s^{(\max)}$ $\qquad \alpha_s^{(\max)} = \dfrac{R_a}{H}$

$$\alpha_s = \begin{cases} \left|\dfrac{r''}{\Delta z}\right| \cdot \alpha_r, & \left|\dfrac{r''}{\Delta z}\right| \cdot \alpha_r < \alpha_s^{(\max)} \\ \alpha_s^{(\max)}, & \text{otherwise} \end{cases}$$

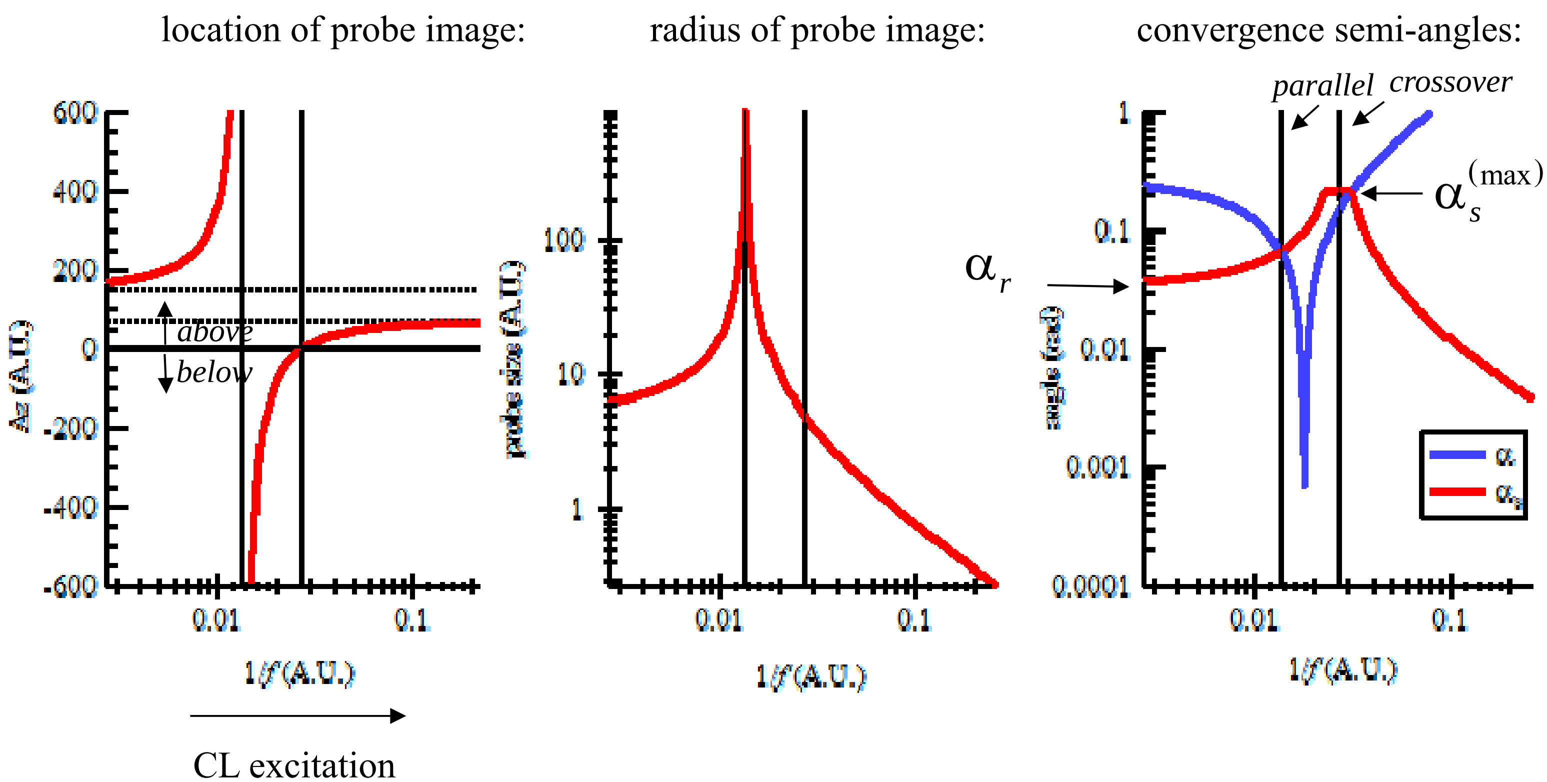
Semi-angle of convergence:

$$\alpha = \left|\frac{R_a}{q_a}\right| = \left|\frac{R_a}{q} - \alpha_r\right|$$

The limiting rays cross the axis at:

$$q_a = \frac{1}{\dfrac{1}{q} - \left(\dfrac{r}{R_a}\right) \cdot \dfrac{1}{p}}$$

Beam convergence: one-lens condenser (III)



Large overfocus is usually better than underfocus.
For underfocus, α_s is never less than α_r .

Beam convergence: two-lens condenser (I)

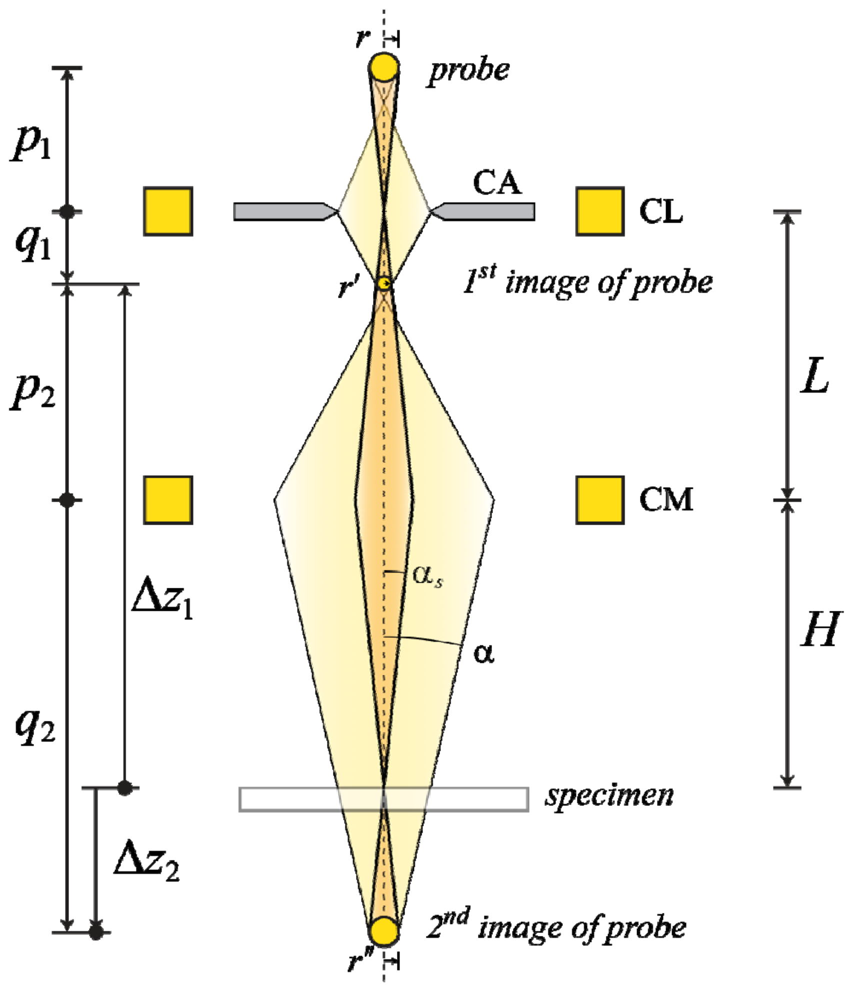
Semi-angle of illumination at a point on the specimen:

$$\alpha_s = \left|\frac{r''}{\Delta z_2}\right| = \frac{\alpha_r}{\left|\left(\frac{L}{q_1}-1\right)\cdot\left(\frac{H}{q_2}-1\right)\right|}$$

$$\alpha_r = r/p_1$$

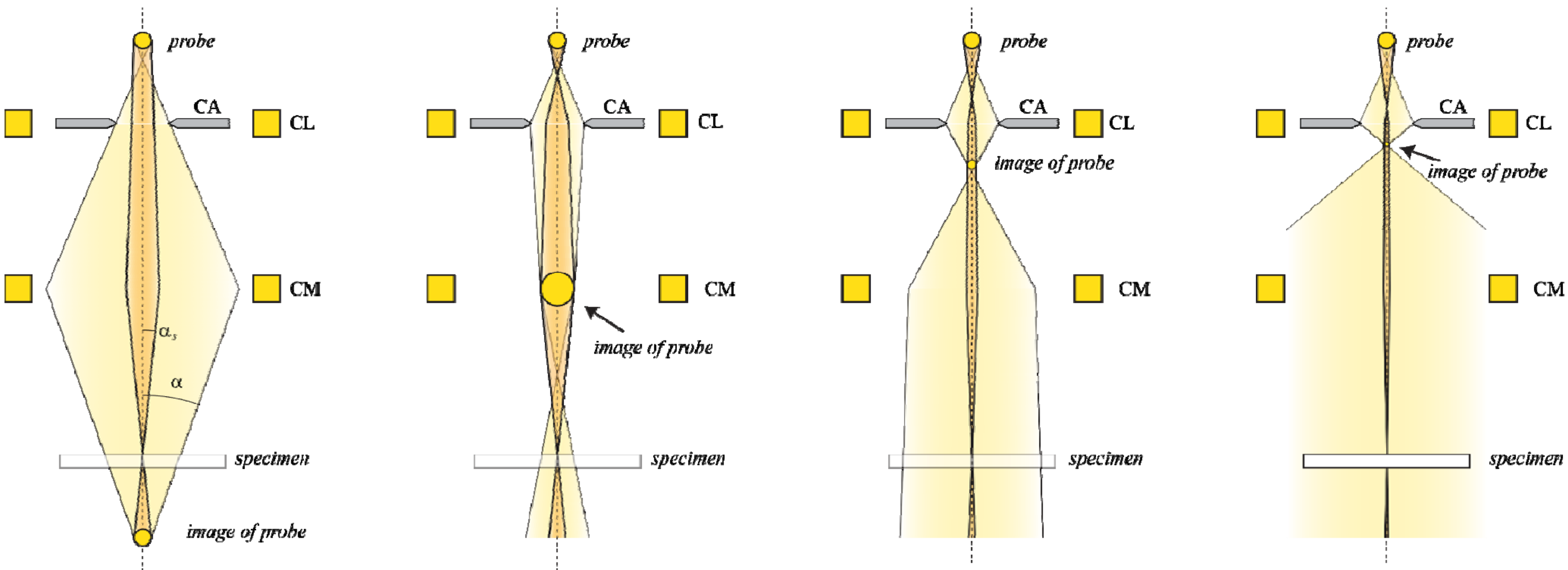
$$\alpha_s^{(\max)} = \frac{R_a}{H\cdot L\cdot\left|\left(\frac{1}{L}-\frac{1}{f_2}-\frac{1}{H}\right)\right|}$$

$$\alpha_s = \begin{cases} \left|\frac{r''}{\Delta z_2}\right|, & \left|\frac{r''}{\Delta z_2}\right|\cdot\alpha_r < \alpha_s^{(\max)} \\ \alpha_s^{(\max)}, & \text{otherwise} \end{cases}$$



Beam convergence: two-lens condenser (II)

Two-lens condenser system:

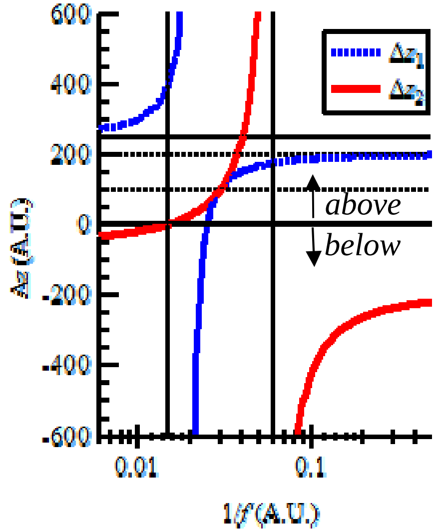


underfocus (CL off) | *CL slightly overfocused* | *nearly parallel* | *nearly optimal (CL highly overfocused)*

Allows small, parallel beam on sample.
Parallel beam formed when probe image formed by CL is at front focal point of CM.
Highly overfocused CL gives small illumination semi-angle on a sample point.

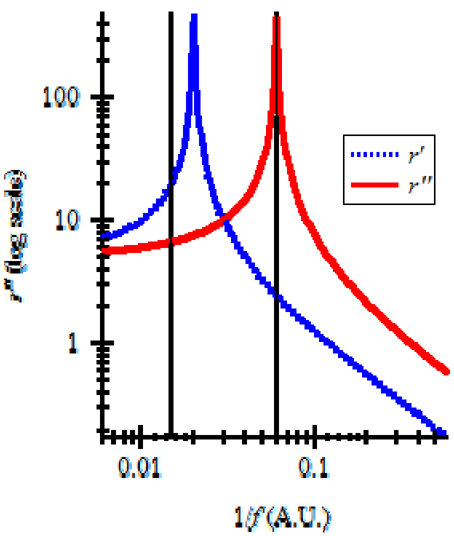
Beam convergence: two-lens condenser (III)

location of probe images:

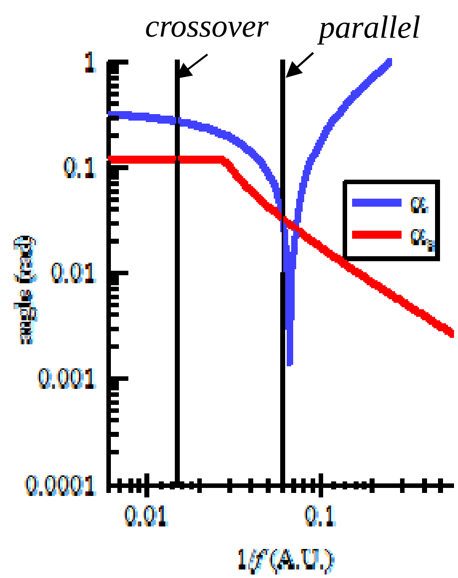


CL excitation

radii of probe images:



convergence semi-angles:



Beyond crossover, α_s decreases with increasing CL3 strength ("Brightness").
Parallel beam formed with CL3 overfocus to specific value; good for diffraction.
Probe size increases with increasing CL3 strength below crossover.

Damping due to beam convergence (I)

Assuming a parallel beam: $G_0\left(u\right) = F_0\left(u\right) \cdot H_0\left(u\right)$

Intensity: $I_0\left(u\right) = \int_{u'=-\infty}^{\infty} du' \cdot G_0^{\,*}\left(u-u'\right) \cdot G_0\left(u'\right)$

For a non-parallel source, we have to integrate over *u*:

Gaussian spot profile: $S\left(u\right) = \dfrac{1}{\sqrt{\pi} k \alpha_s} \cdot \mathrm{e}^{-\left(\frac{u}{k\alpha_s}\right)^2}$

There are two distinct ways to combine the incidence angles: coherently and incoherently

coherent: $G\left(u\right) = \int_{u'=-\infty}^{\infty} du' \cdot G_0\left(u-u'\right) \cdot S\left(u'\right)$

incoherent: $I\left(u\right) = \int_{u'=-\infty}^{\infty} du' \cdot I_0\left(u-u'\right) \cdot S\left(u'\right)$

Damping due to beam convergence (II)

Assume only effect is a phase shift : $G_0(u) = F(u) \cdot \mathrm{e}^{-i\chi(u)}$

Let's assume the different incidence angles combine coherently.

$$G(u) = \frac{1}{\sqrt{\pi} k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot F(u-u') \cdot \mathrm{e}^{-i\chi(u-u')} \cdot \mathrm{e}^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

If the input is a delta function, the output is the transfer function:

$$H(x) = S[\delta(x)]$$

$$F(x) = \delta(x) \rightarrow F(u) = 1$$

$$H(u) = \frac{1}{\sqrt{\pi} k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot \mathrm{e}^{-i\chi(u-u')} \cdot \mathrm{e}^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

Damping due to spatial incoherence (II)

Assume $k\alpha_s$ is small.

$$\chi(u-u') \approx \chi(u) - u' \cdot \left.\frac{\partial\chi(u'')}{\partial u''}\right|_{u''=u} = \chi(u) - u' \cdot C(u)$$

Do the integral:

$$H(u) \approx \mathrm{e}^{-i\chi(u)} \cdot \frac{1}{\sqrt{\pi}k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot \mathrm{e}^{-iC(u)\cdot u'} \cdot \mathrm{e}^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

$$= \mathrm{e}^{-i\chi(u)} \cdot \frac{1}{\sqrt{\pi}k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot \cos\left[C(u) \cdot u'\right] \cdot \mathrm{e}^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

$$H(u) = \mathrm{e}^{-i\chi(u)} \cdot \exp\left[-\left(\frac{k\alpha_s}{2} \cdot C(u)\right)^2\right] = \mathrm{e}^{-i\chi(u)} \cdot E_s(u)$$

Spatial incoherence is represented by a damping function:

$$E_s(u) = \exp\left[-\left(\frac{k\alpha_s}{2} \cdot \frac{\partial\chi}{\partial u}\right)^2\right]$$

Combine incoherence effects

We can combine temporal and spatial incoherence functions:

$$E(u) = E_c(u) \cdot E_s(u) \cdots$$

Uniform-Field Electron Lens (I)

$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$ //canonical momentum

$\mathbf{B} = \nabla \times \mathbf{A} = B_\rho \hat{\boldsymbol{\rho}} + B_z \hat{\mathbf{z}}$ //magnetic field in axially symmetric lens

$B_\rho = -\frac{\rho B_0}{2} \cdot \delta(z+a) + \frac{\rho B_0}{2} \delta(z-a)$ //uniform-field model

$B_z = B_0 \cdot [u(z+a) - u(z-a)]$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \cdot \hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \cdot \hat{\boldsymbol{\phi}} + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \cdot \hat{\mathbf{z}}$$

$\mathbf{A} = \frac{\rho B_0}{2} \cdot [u(z+a) - u(z-a)] \hat{\boldsymbol{\phi}}$ //magnetic vector potential

Uniform-Field Electron Lens (II)

Assume: $\dot{\phi} = 0 \;\; (z < -a)$

$$\mathbf{v} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

$$\begin{aligned}\mathbf{v} = &-k' \cdot v_z' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\rho}} \\ &+\rho_0 \cdot \omega_L \cdot \cos[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\phi}} \\ &+v_z'\hat{\mathbf{z}}\end{aligned}$$

$$\dot{\phi} = \omega_L$$

$$\rho = \rho_0 \cdot \cos[k' \cdot (z+a) + \theta]$$

$$\dot{\rho} = -k' \cdot v_z' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta]$$

$$\dot{z} = v_z'$$

$$\begin{aligned}\mathbf{p} &= m\mathbf{v} + q\mathbf{A} = m\mathbf{v} - e\mathbf{A} \\ &= -m \cdot k' \cdot v_z' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\rho}} \\ &\quad +m \cdot \omega_L \cdot \rho_0 \cdot \cos[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\phi}} \\ &\quad +m \cdot v_z'\hat{\mathbf{z}} \\ &\quad -\frac{e \cdot B_0 \cdot \rho_0}{2} \cdot \cos[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\phi}}\end{aligned}$$

$$\omega_L = \frac{eB_0}{2m} \qquad k' = \frac{\omega_L}{v_z'}$$

$$v_z' \equiv \sqrt{{v_z}^2 - (\rho_0\omega_L)^2}$$

$$\mathbf{p} = m \cdot v_z' \cdot \left\{-k' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}}\right\}$$

Uniform-Field Electron Lens (III)

$\psi(\mathbf{r}) = e^{i\phi(\mathbf{r})}$ //electron wave function

$\hat{\mathbf{p}} = \frac{h}{i}\nabla$ //momentum operator

$\hat{\mathbf{p}}\psi = [h \cdot \nabla\phi(\mathbf{r})] \cdot \psi(\mathbf{r})$

$\nabla\phi(\mathbf{r}) = \mathbf{p}/h$ //momentum proportional to gradient of phase

$\delta\phi(\mathbf{r}) = \frac{2\pi}{h} \cdot \int_{\mathbf{r}=\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{p} \cdot d\mathbf{r} = 2\pi \cdot \delta n(\mathbf{r})$ // phase change

$\delta n(\mathbf{r}) = \frac{1}{h} \cdot \int_{\mathbf{r}=\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{p} \cdot d\mathbf{r}$ //change in number of wave fronts

$\mathbf{p} = m \cdot v_z' \cdot \{-k' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta]\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}}\}$ //momentum in uniform-field lens

$d\mathbf{r} = d\rho \cdot \hat{\boldsymbol{\rho}} + \rho \cdot d\phi \cdot \hat{\boldsymbol{\phi}} + dz \cdot \hat{\mathbf{z}}$

$\mathbf{p} \cdot d\mathbf{r} = m \cdot v_z' \cdot \{-k' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta] \cdot d\rho + dz\}$ //needed to find phase change

Electron Lenses (III)

$$d\rho = -k' \cdot \rho_0 \cdot \sin[k' \cdot (z+a) + \theta] \cdot dz$$

$$\mathbf{p} \cdot d\mathbf{r} = m \cdot v_z' \cdot \left\{ (k' \cdot \rho_0)^2 \cdot \sin^2[k' \cdot (z+a) + \theta] + 1 \right\} \cdot dz$$ //differential change

$$\delta n_{\rho_0,\theta}(z) = \frac{m \cdot v_z'}{h} \cdot \int_{z'=-a}^{z} \left\{ (k' \cdot \rho_0)^2 \cdot \sin^2[k' \cdot (z'+a) + \theta] + 1 \right\} \cdot dz$$

$$u \equiv k \cdot \rho_0 \qquad \rho_0 \cdot \omega_L = \rho_0 \cdot (k \cdot v_z) = u \cdot v_z \qquad v_z' \equiv v_z \cdot \sqrt{1-u^2}$$

$$\lambda = \frac{h}{m \cdot v} \qquad \lambda_z = \frac{h}{m \cdot v_z} \qquad \lambda_z' = \frac{h}{m \cdot v_z'} = \lambda_z \Big/ \sqrt{1-u^2}$$

$$\delta n_{\rho_0,\theta}(z) \cdot \lambda_z' = \int_{z'=-a}^{z} \left\{ (k' \cdot \rho_0)^2 \cdot \sin^2[k' \cdot (z'+a) + \theta] + 1 \right\} \cdot dz$$ //# of wave fronts along z

Uniform-Field Electron Lens (IV)

$$x' \equiv k' \cdot z$$

$$\delta n_{\rho_0,\theta}(z) \cdot \frac{\lambda'_z}{a} = \left(\frac{1}{k'a}\right) \int_{x'=-k'a}^{k'z} \left\{ (k' \cdot \rho_0)^2 \cdot \sin^2[x' + k' \cdot a + \theta] + 1 \right\} \cdot dx'$$

$$= \left(\frac{1}{k'a}\right) \left\{ (k' \cdot \rho_0)^2 \cdot \left[\frac{x' + k' \cdot a + \theta}{2} - \frac{\sin[2(x + k' \cdot a + \theta)]}{4} \right] + x' \right\} \Bigg|_{x=-k'a}^{k'z}$$

$$\delta n_{\rho_0,\theta}(z) \cdot \frac{\lambda'_z}{a} = \left[\frac{(k' \cdot \rho_0)^2}{2} + 1 \right] \cdot \left(\frac{z}{a} + 1 \right) - \frac{(k' \cdot \rho_0)^2}{4k'a} \left(\sin\left\{ 2\left[k'a \cdot \left(\frac{z}{a} + 1 \right) + \theta \right] \right\} - \sin(2\theta) \right)$$

ρ_i: radius at $z = -a$

θ_i: angle w.r.t. optic axis at $z = -a$

$$\rho_0 = \sqrt{\rho_i^{\,2} + \frac{\tan^2 \theta_i}{k'^2}}$$

$$\theta = -\tan^{-1}\left(\frac{\tan \theta_i}{k' \cdot \rho_i} \right)$$

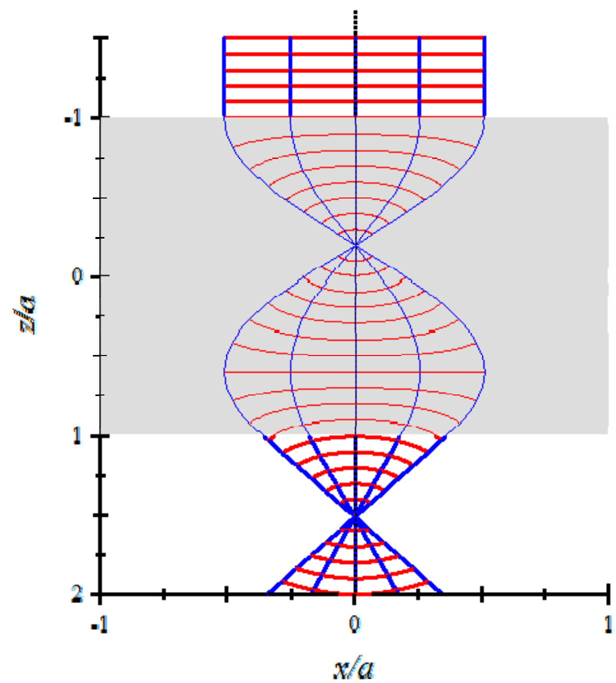
Uniform-Field Electron Lens

$u^2 \ll 1$ Expand to lowest order in *u*:

$$\delta n_{u,\theta}(z)\cdot\frac{\lambda_z}{a}=\frac{z}{a}+1-\frac{u^2}{4ka}\left(\sin\left\{2\left[ka\cdot\left(\frac{z}{a}+1\right)+\theta\right]\right\}-\sin(2\theta)\right)$$

$\theta_i = 0$ $\rho_i = \rho_0$

$$\delta n_u(z)\cdot\frac{\lambda_z}{a}=\frac{z}{a}+1-\frac{u^2}{4ka}\cdot\sin\left[2ka\cdot\left(\frac{z}{a}+1\right)\right]$$



In this limit, the lens has no spherical aberration

Image of periodic specimen (I)

Object function:

$$F(x) = \sum_g F_g \cdot \mathrm{e}^{2\pi igx}$$

$$F(u) = \Im\{F(x)\}$$
$$= F_g \cdot \Im\left\{\sum_g \mathrm{e}^{2\pi igx}\right\}$$
$$F(u) = \sum_g F_g \cdot \Delta(u-g)$$

Transfer function (no attenuation):

$$H(u) = A(u) \cdot \mathrm{e}^{-i\chi(u)}$$

$$G(u) = \left[\sum_g F_g \cdot \varDelta(u-g)\right] \cdot A(u) \cdot \mathrm{e}^{-i\chi(u)}$$

Image function:

$$G(x) = \lim_{K\to\infty}\left[\int_{u=-K}^{K} G(u) \cdot \mathrm{e}^{2\pi iux} \cdot du\right]$$
$$= \lim_{K\to\infty}\left\{\int_{u=-K}^{K} \sum_g F_g \cdot \Delta(u+g) \cdot A(u) \cdot \mathrm{e}^{-i\chi(u)} \cdot \mathrm{e}^{2\pi iux} \cdot du\right\}$$
$$G(x) = \sum_g F_g \cdot A(g) \cdot \mathrm{e}^{-i\chi(g)} \cdot \mathrm{e}^{2\pi igx}$$

Image of periodic specimen (II)

FT of image function:

$$G(u) = \lim_{L\to\infty}\left[\int_{x=-L}^{L} G(x)\cdot \mathrm{e}^{-2\pi i u x}\cdot dx\right]$$

$$= \lim_{L\to\infty}\int_{x=-L}^{L}\left\{\sum_g \left[F_g \cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\cdot \mathrm{e}^{2\pi i g x}\right]\cdot \mathrm{e}^{-2\pi i u x}\cdot dx\right\}$$

$$= \sum_g \left\{\lim_{L\to\infty}\int_{x=-L}^{L}\left[F_g \cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\cdot \mathrm{e}^{2\pi i g x}\right]\cdot \mathrm{e}^{-2\pi i u x}\cdot dx\right\}$$

$$= \sum_g F_g \cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\left\{\lim_{L\to\infty}\int_{x=-L}^{L}\mathrm{e}^{-2\pi i (g-u)x}\cdot dx\right\}$$

$$= \sum_g F_g \cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\varDelta(u-g)$$

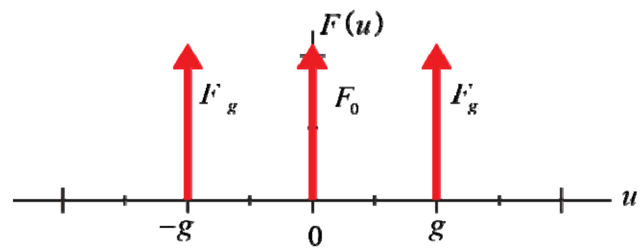
$$G(u) = \sum_g G_g \cdot \varDelta(u-g)$$

$$G_g = F_g \cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}$$

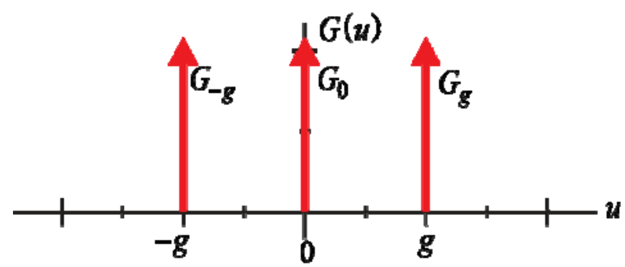
$$G(x) = \sum_g G_g \cdot \mathrm{e}^{2\pi i g x}$$

Image of periodic specimen (III)

Three-beam case:



$$F(u) = F_{-g} \cdot \Delta(u+g) + F_0 \cdot \Delta(u) + F_g \cdot \Delta(u-g)$$



$$G(u) = G_{-g} \cdot \Delta(u+g) + G_0 \cdot \Delta(u) + G_g \cdot \Delta(u-g)$$

	$-g$	0	g
$F(u)$	F_{-g}	F_0	F_g
$A(u)$	$A(-g)$	$A(0)$	$A(g)$
$e^{-i\chi(u)}$	$e^{-i\chi(-g)}$	$e^{-i\chi(0)}$	$e^{-i\chi(g)}$
$G(u)$	$F_{-g} \cdot A(-g) \cdot e^{-i\chi(-g)}$	$F_0 \cdot A(0) \cdot e^{-i\chi(0)}$	$F_g \cdot A(g) \cdot e^{-i\chi(g)}$

Example

Given: $F(x) = 1 + iB \cdot \sin(2\pi g x)$ $\quad A(u) = 1 \quad$ $\chi(u) = \begin{cases} 0, & |u| \le g/2 \\ -\pi/2, & g/2 < |u| \end{cases}$

Write: $F(x) = -\dfrac{B}{2}\mathrm{e}^{-2\pi i g x} + 1 + \dfrac{B}{2}\mathrm{e}^{2\pi i g x} = F_{-g}\mathrm{e}^{-2\pi i g x} + F_0 + F_g \mathrm{e}^{2\pi i g x}$

	$-g$	0	g
$F(u)$	$-B/2$	1	$B/2$
$A(u)$	1	1	1
$\chi(u)$	$-\pi/2$	0	$-\pi/2$
$\mathrm{e}^{-i\chi(u)}$	i	1	i
$G(u)$	$-iB/2$	1	$iB/2$

$\Rightarrow$ $G(x) = -\dfrac{iB}{2}\mathrm{e}^{-2\pi i g x} + 1 + \dfrac{iB}{2}\mathrm{e}^{2\pi i g x} = 1 - B\sin(2\pi g x)$