Linear Systems

Describes the output of a linear system

$$G(x) = \int_{x'} F(x') \cdot H(x - x') \cdot dx' = F(x) * H(x)$$

output *input*

Impulse response function

$$H(x) = \int_{x'} \delta(x') \cdot H(x - x') \cdot dx'$$

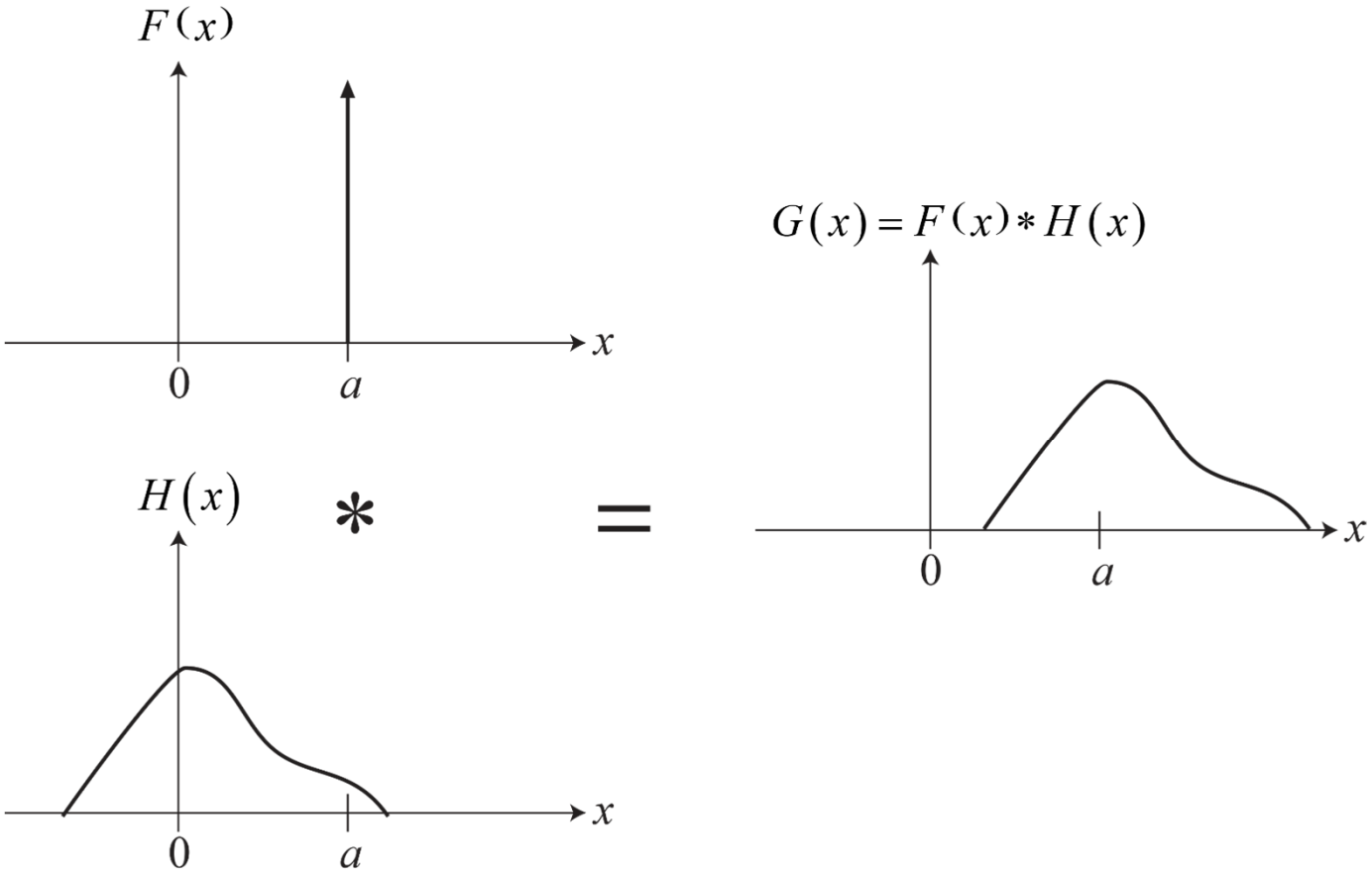
If the microscope is a linear system:

$F(x)$: object

$G(x)$: image

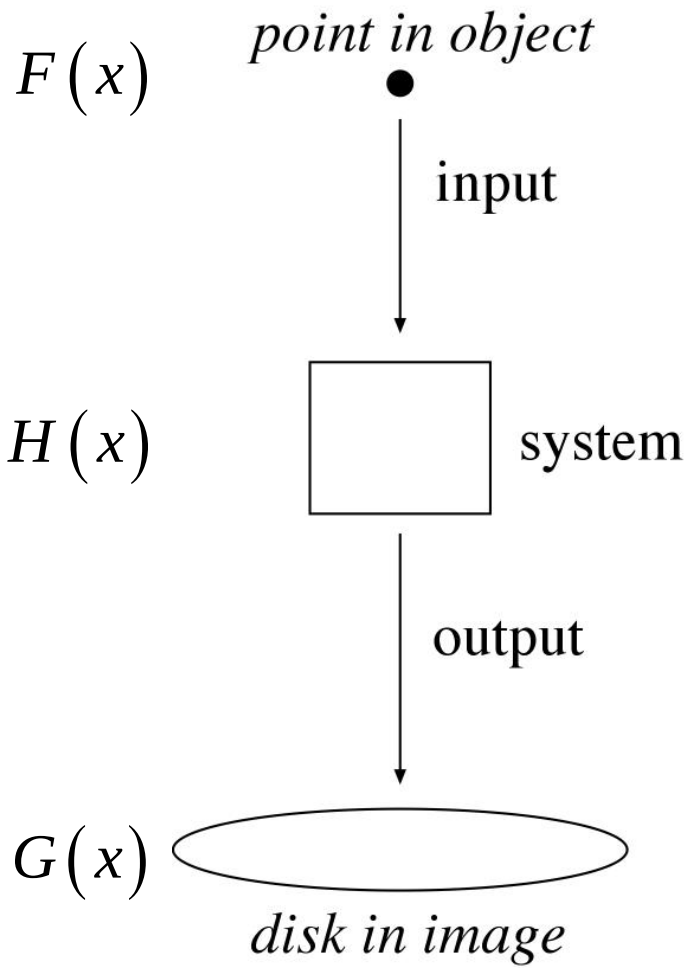
$$G(x) = S[F(x)]$$

Transfer Function



$H(x)$ is also called the "transfer function" of the system

Microscope as a Linear System



Convolution in direct space:

$$G(x) = F(x) * H(x)$$

Fourier Transforms:

$$F(u) = \Im\{F(x)\}$$

$$G(u) = \Im\{G(x)\}$$

$$H(u) = \Im\{H(x)\}$$

Convolution Theorem:

→Multiplication in reciprocal space

$$G(u) = F(u) \cdot H(u)$$

Contributions to $H(u)$

$$H(u) = A(u) \cdot E(u) \cdot \mathrm{e}^{-i\chi(u)}$$

$A(u)$: aperture function

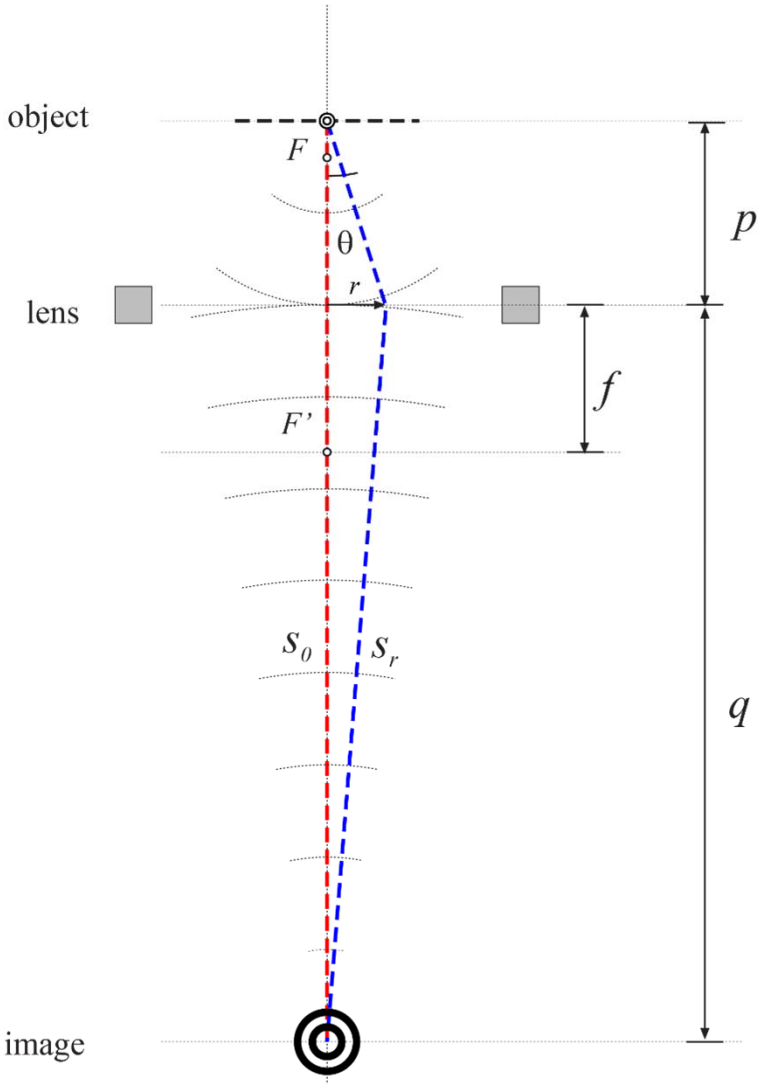
$E(u)$: envelope (damping) function

$\chi(u)$: aberration (phase) function

Most important for phase-contrast imaging:

We need to find $\chi(u)$:

Path Length Correction due to Lens



Ideal lens: $$\frac{1}{p}+\frac{1}{q}=\frac{1}{f_0}$$

$$s_r=\sqrt{p^2+r^2}+\sqrt{q^2+r^2}+\Delta s(r)$$

$$\approx\left(p+\frac{r^2}{2p}\right)+\left(q+\frac{r^2}{2q}\right)+\Delta s(r)$$

$$=p+q+\frac{r^2}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\Delta s(r)$$

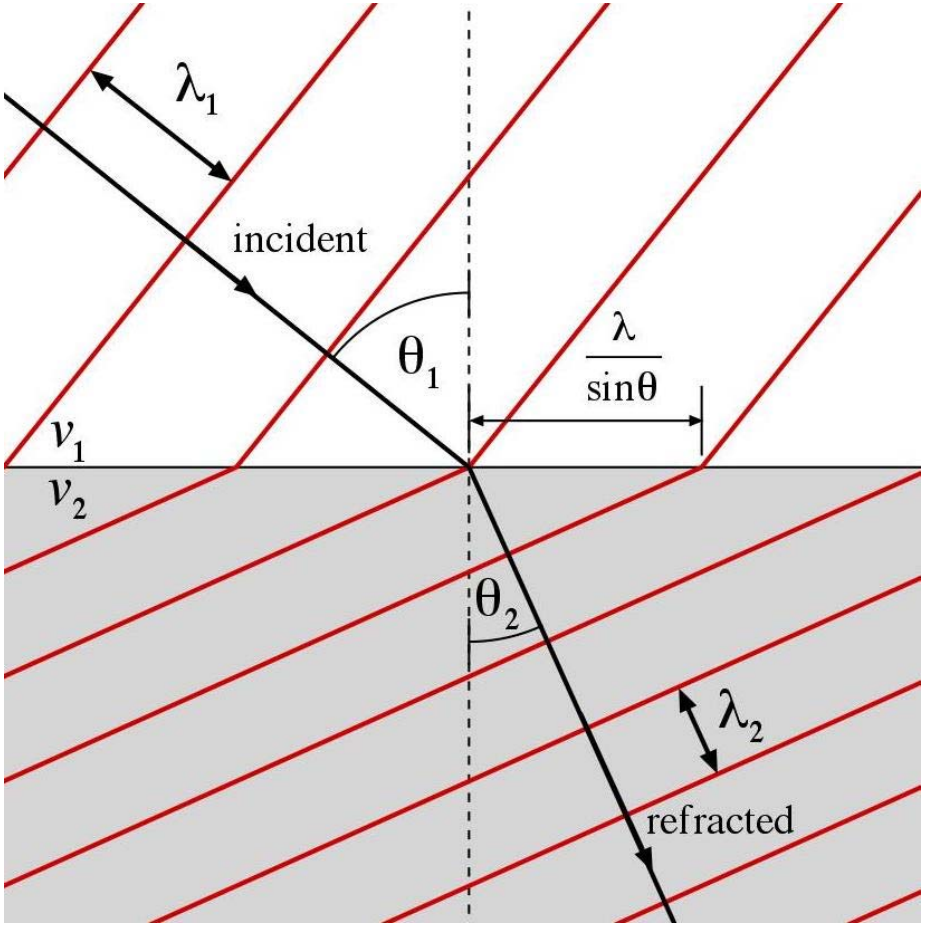
$$s_r=p+q+\frac{r^2}{2f_0}+\Delta s(r)$$

We expect the same net path length for all focused rays:

$$s_r=s_0=p+q \qquad \Rightarrow \Delta s(r)=-\frac{r^2}{2f_0}$$

Snell's Law

The wave crests must be continuous across the interface:



$$\frac{\sin\theta_1}{\lambda_1} = \frac{\sin\theta_2}{\lambda_2}$$

$$\frac{c}{f}\cdot\left(\frac{\sin\theta_1}{\lambda_1} = \frac{\sin\theta_2}{\lambda_2}\right)$$

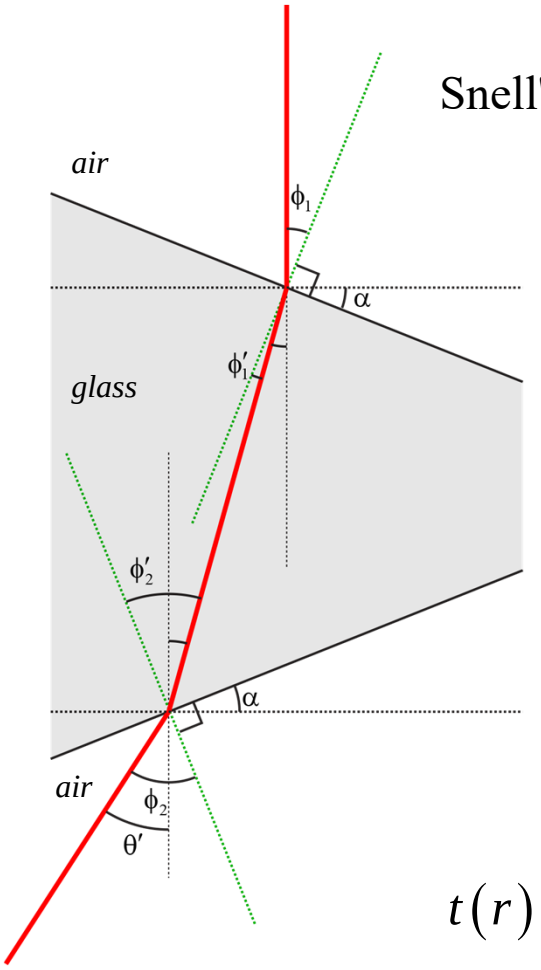
$$v_1 = f\cdot\lambda_1 = c/n_1$$
$$v_2 = f\cdot\lambda_2 = c/n_2$$

refractive indices

$$\Rightarrow \quad n_1\cdot\sin\theta_1 = n_2\cdot\sin\theta_2$$

This applies to any type of wave (not only light).

Thin Optical Lens (I)



Snell's law: $\sin\phi_1 = n\cdot\sin\phi_1'$

$\sin\phi_2 = n\cdot\sin\phi_2'$

$\phi_1 = \alpha$

$\phi_1' + \phi_2' = 2\alpha$

Small angles: $\phi_1 \approx n\phi_1'$

$\phi_2 \approx n\phi_2'$

$\theta' = \phi_2 - \alpha$

$$\theta' = n\phi_2' - \alpha = n\left(2\alpha - \phi_1'\right) - \alpha = 2(n-1)\alpha$$

$$\tan\theta' = \frac{r}{f_0} \approx \theta' \qquad \alpha = \frac{r}{2(n-1)\,f_0}$$

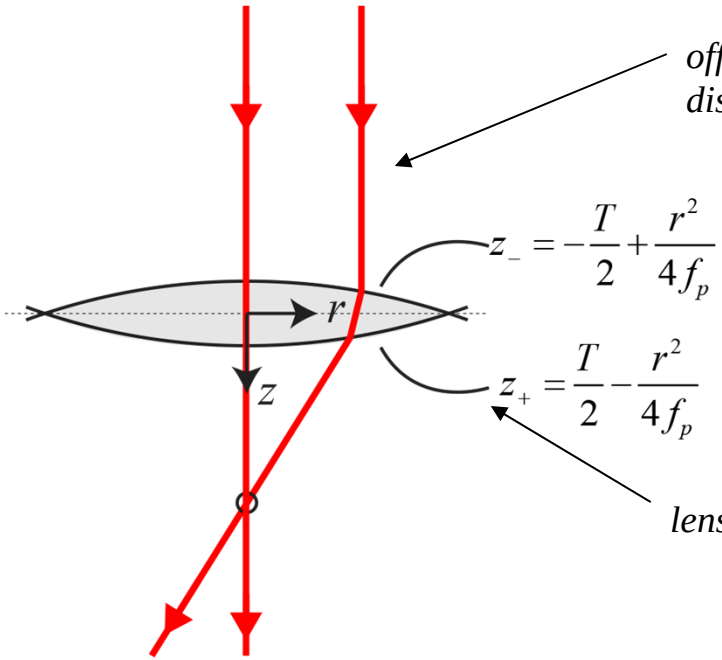
$$\frac{dt}{dr} = -2\tan\alpha \approx -2\alpha = -\frac{r}{(n-1)\,f_0}$$

$$t(r) = T - \int_{r'=0}^{r} \frac{r'\cdot dr'}{(n-1)\,f_0} = T - \frac{r^2}{2(n-1)\,f_0} = T - \Delta t(r)$$

Thickness change: $\Delta t(r) = \dfrac{r^2}{2(n-1)\,f} = \dfrac{r^2}{2\,f_p}$ ← parabola focal length

Optical focal Length: $f_0 = \dfrac{f_p}{n-1}$

Thin Optical Lens (II)



off-axis rays have a greater distance to travel

lens thinner at edges

Wavelength:

vacuum: $\lambda_0 = c/f$

medium: $\lambda = \frac{c}{nf} = \lambda_0/n$

Phase difference: $\Delta\phi(r) = 2\pi \cdot [T - t(r)] \cdot \left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$

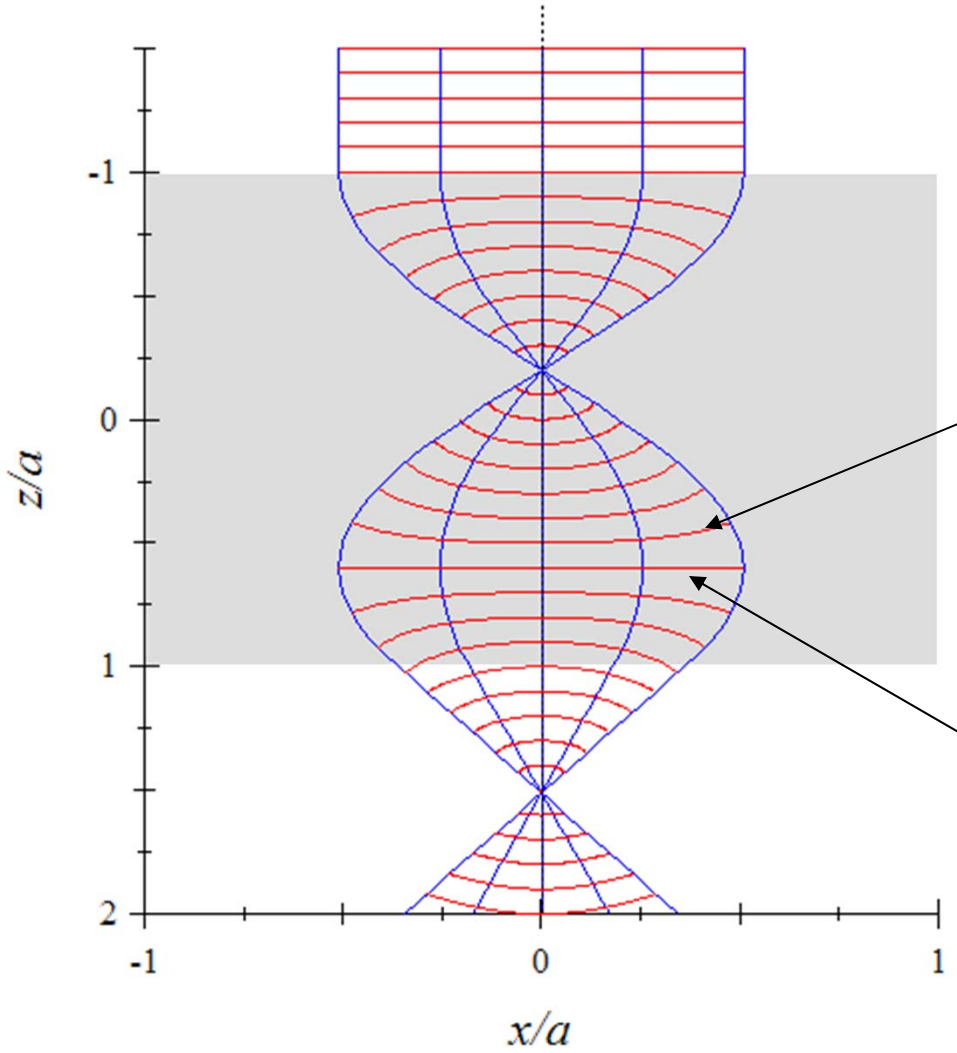
$$= -2\pi \cdot \frac{(n-1)\cdot \Delta t(r)}{\lambda_0}$$

$$\Delta\phi(r) = 2\pi \cdot \left(\frac{\Delta s(r)}{\lambda_0}\right)$$

$$\frac{d(\Delta t)}{dr} = \frac{r}{(n-1)\, f_0} \qquad \Delta s(r) = -(n-1)\cdot \Delta t(r) \qquad \frac{d(\Delta s)}{dr} = -\frac{r}{f_0}$$

Optical path-length difference: $\Delta s(r) = -\int_{r'=0}^{r} \frac{r' \cdot dr'}{f_0} = -\frac{r^2}{2 f_0}$

Uniform-Field Electron Lens



off-axis rays have a greater distance to travel

less vertical momentum, longer wavelength

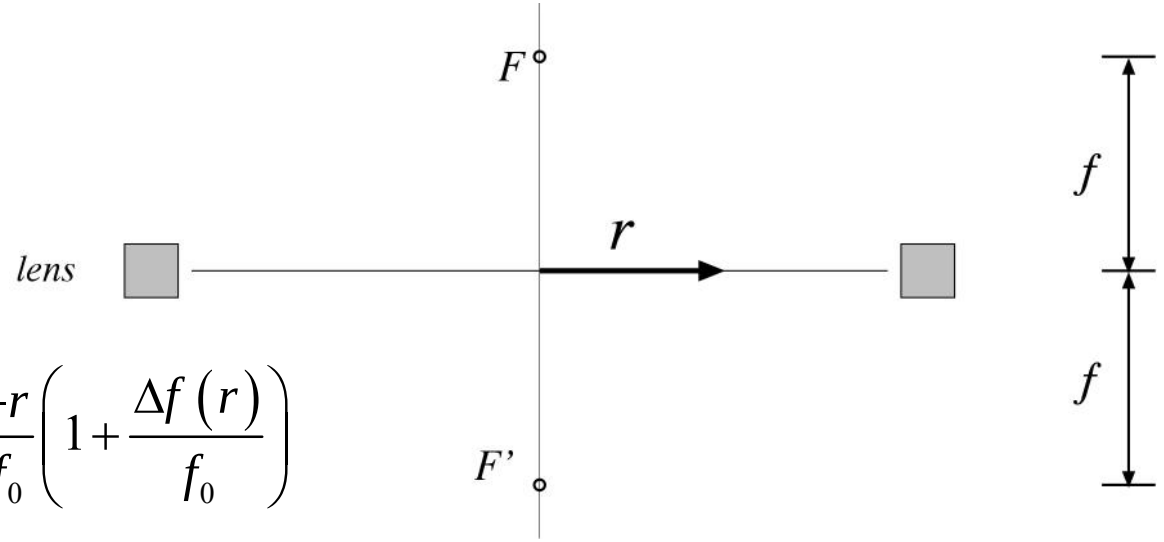
Path Length Correction due to Lens

A path-length correction for each trajectory through the lens should be included.

Generalize: $$\frac{d(\Delta s)}{dr} = \frac{-r}{f_0} \rightarrow \frac{-r}{f(r)}$$

Non-ideal lens:

$$f(r) = f_0 - \Delta f(r)$$



$$\frac{d(\Delta s)}{dr} = \frac{-r}{f_0 - \Delta f(r)} \approx \frac{-r}{f_0}\left(1 + \frac{\Delta f(r)}{f_0}\right)$$

$$\Delta s(r) = \frac{-r^2}{2 f_0} - \int_{r'=0}^{r}\left[\frac{r' \cdot \Delta f(r')}{f_0^{\,2}}\right] \cdot dr'$$

The derivative of the path-length difference determines the focal length.

Influence of Spherical Aberration

From Chap. 6: $$\Delta f(r) = C_s \cdot \left(\frac{r}{f_0}\right)^2$$

$$\Delta s(r) = \frac{-r^2}{2 f_0} - \int_{r'=0}^{r} \left(\frac{r'}{f_0{}^2}\right) \cdot \left[C_s \cdot \left(\frac{r'}{f_0}\right)^2 \right] dr'$$

$$= \frac{-r^2}{2 f_0} - \frac{1}{4} C_s \left(\frac{r}{f_0}\right)^4$$

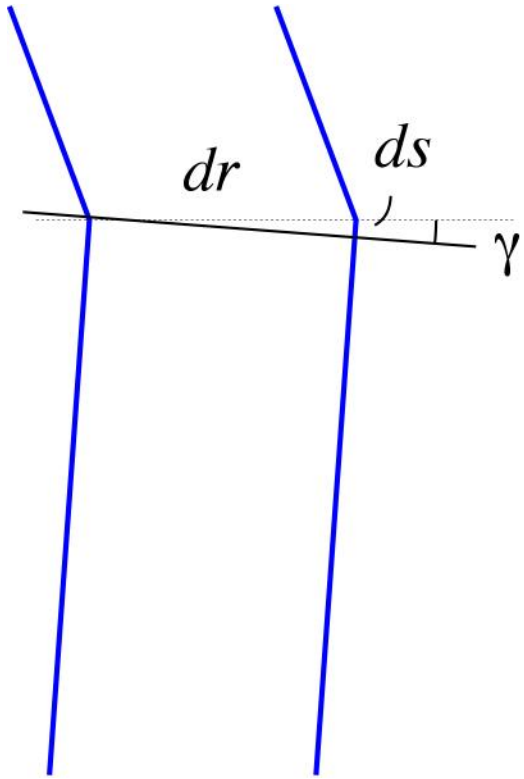
normal focusing

aberration

Spherical aberration (wave optics)

When object is at focal point, if no aberration,
all rays on image side (back) of lens will be parallel to lens

With aberration, off-axis rays will be tend toward the optic axis

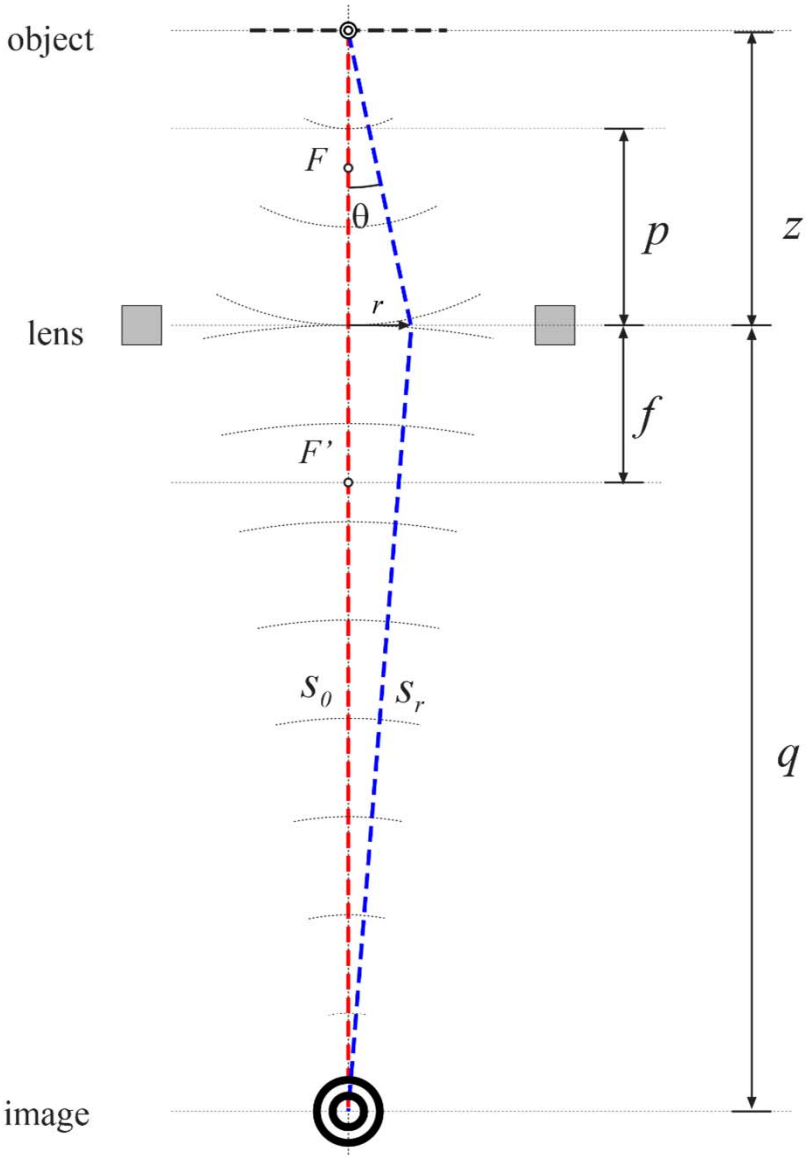


$$\gamma = \frac{C_s r^3}{f_0^{\,4}} \qquad \gamma = \frac{ds}{dr}$$

$$ds = \gamma \cdot dr = \frac{C_s r^3}{f_0^{\,4}} \cdot dr$$

$$\Delta s(r) = \frac{C_s}{f_0^{\,4}} \int_{r'=0}^{r} r'^3 \cdot dr' = \frac{C_s r^4}{4 f_0^{\,4}}$$

Path-length difference: general case



Assume object is on-axis, but not in focus.

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f_0}$$

Axial Ray: $s_0 = z + q$

Lens: $$\Delta s(r) = \frac{-r^2}{2f} - \frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4$$

Off-axis Ray:

$$s_r = \sqrt{z^2+r^2} + \sqrt{q^2+r^2} + \Delta s(r)$$

$$\approx z + \frac{r^2}{2z} + q + \frac{r^2}{2q} + \Delta s(r)$$

$$s_r = s_0 + \Delta s_{net}(r)$$

$$\Delta s_{net}(r) = \frac{r^2}{2}\cdot\left(\frac{1}{z}+\frac{1}{q}-\frac{1}{f}\right) - \frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4$$

Combine defocus and sample height

$$p = z + \Delta z$$

$$f = f_0 - \Delta f$$

$$\frac{1}{z} + \frac{1}{q} - \frac{1}{f} = \frac{1}{p - \Delta z} + \frac{1}{q} - \frac{1}{f_0 - \Delta f}$$

$$\approx \cancel{\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{f_0} \right)} + \frac{\Delta z}{p^2} - \frac{\Delta f}{f_0^{\,2}}$$

$$\approx \frac{\Delta z}{p^2} - \frac{\Delta f}{f_0^{\,2}}$$

$$M_0 = \frac{q}{p} = \frac{q}{f_0} - 1 \approx \frac{q}{f_0} \gg 1 \rightarrow f_0 \ll q$$

$$p \approx f_0$$

$$\frac{1}{z} + \frac{1}{q} - \frac{1}{f} \approx \frac{\Delta z - \Delta f}{f_0^{\,2}}$$

only the difference matters

Optical path-length difference

$$\Delta s_{net} \approx \frac{1}{2}\cdot(\Delta z - \Delta f)\cdot\left(\frac{r}{f_0}\right)^2 - \frac{1}{4}C_s\left(\frac{r}{f_0}\right)^4$$

$$r = z\cdot\theta$$

$$\Delta s = \frac{1}{2}\cdot(\Delta z - \Delta f)\cdot\left(\frac{z}{f_0}\right)^2\cdot\theta^2 - \frac{1}{4}C_s\left(\frac{z}{f_0}\right)^4\cdot\theta^4$$

$$\frac{z+\cancel{\Delta z}}{f_0} = \frac{1}{1-f_0/q} \approx 1+\cancel{f_0/q} \rightarrow z \approx f_0$$

Net Effect: $$\Delta s \approx \frac{1}{2}\cdot(\Delta z - \Delta f)\cdot\theta^2 - \frac{1}{4}C_s\cdot\theta^4$$

Phase correction

Phase shift: $$\Delta\phi = -2\pi\left(\frac{\Delta s}{\lambda}\right) = \frac{\pi}{\lambda}\cdot\left[\left(\Delta f - \Delta z\right)\cdot\theta^2 + \frac{1}{2}C_s\theta^4\right]$$

Scattering angle: $$\theta = \frac{R}{L} = \frac{\lambda}{d} = \lambda u$$

Frequency representation: $$\chi\left(u\right) = \pi\cdot\left(\Delta f - \Delta z\right)\cdot\lambda u^2 + \frac{\pi}{2}C_s\lambda^3 u^4$$

Image function for weak-phase object

A weak phase object produces an image function:

$$G(x) = F(x) * H(x)$$

$$G(x) = \left[1 + i\sigma V_t(x)\right] * H(x)$$

First term: $\Im\{1 * H(x)\} = \Delta(u) \cdot H(u)$

$$\Im^{-1}\{\Delta(u) \cdot H(u)\} = H(u)\big|_{u=0}$$

The overall phase doesn't matter, so we might as well pick

$$H(u)\big|_{u=0} = 1$$

$$G(x) \approx 1 + i\sigma V_t(x) * H(x)$$

Simple situation

We might expect the ideal transfer function to be:

$$H(x) = \delta(x)$$

$$G(x) = \left[1 + i\sigma V_t(x)\right] * H(x) = 1 + i\sigma V_t(x)$$

Intensity:

$$I(x) = \left|G(x)\right|^2 = 1 + \cancel{\left[\sigma V_t(x)\right]^2} \approx 1$$

This gives no contrast to first order in σ

We will need a relative phase shift between the incident and scattered beams to see phase contrast

Contrast transfer function

Intensity:

$$
\begin{aligned}
I(x) &= \left|G(x)\right|^2 \\
&= \left[1 - i\sigma V_t(x) * H^*(x)\right] \cdot \left[1 + i\sigma V_t(x) * H(x)\right] \\
&\approx 1 + i\sigma V_t(x) * \left[H(x) - H^*(x)\right] \\
&= 1 - 2\sigma V_t(x) * \mathrm{Im}\left[H(x)\right] \\
I(x) &= 1 - \sigma V_t(x) * T(x)
\end{aligned}
$$

$$
T(x) \doteq 2\,\mathrm{Im}\left[H(x)\right]
$$

CTF in reciprocal space

$$I(x) = 1 - \sigma V_t(x) * T(x)$$

$$I(u) = \Im\left[1 - \sigma V_t(x) * T(x)\right] = \Delta(u) - \sigma V_t(u) \cdot T(u)$$

$$\begin{aligned}
T(u) &= \Im[T(x)] \\
&= \Im\left\{2 \cdot \mathrm{Im}\left[H(x)\right]\right\} \\
&= -i \cdot \Im\left[H(x) - H^*(x)\right] \\
&= -i \cdot \int_x dx \cdot \left\{\int_{u'} du' \cdot \left[H(u') \cdot \mathrm{e}^{2\pi i u' x} - H^*(u') \cdot \mathrm{e}^{-2\pi i u' x}\right]\right\} \cdot \mathrm{e}^{-2\pi i u x} \\
&= -i \cdot \int_{u'} du' \cdot \left\{H(u') \cdot \left[\int_x dx \cdot \mathrm{e}^{2\pi i (u'-u) x}\right] - H^*(u') \cdot \left[\int_x dx \cdot \mathrm{e}^{-2\pi i (u'+u) x}\right]\right\} \\
&= -i \cdot \int_{u'} du' \cdot \left\{H(u') \cdot \Delta(u'-u) - H^*(u') \cdot \Delta(u'+u)\right\}
\end{aligned}$$

$$T(u) = -i \cdot \left[H(u) - H^*(-u)\right]$$

Special Case: $H(u)$ is an even function

If $H(-u) = H(u)$

then $H^*(-u) = H^*(u)$

$$T(u) = -i \cdot \left[H(u) - H^*(-u) \right]$$

$$= -i \left[H(u) - H^*(u) \right]$$

$$T(u) = 2 \cdot \mathrm{Im}\left[H(u) \right]$$

Using the CTF

$$H(u) = A(u) \cdot E(u) \cdot \exp\left[-i\chi(u)\right]$$

Assume: $H(u) = H(-u)$

$$T(u) = 2\,\mathrm{Im}\left[H(u)\right] = -2A(u)E(u)\sin\chi(u)$$

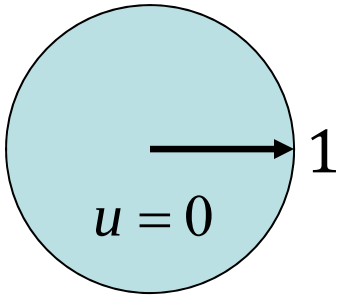
$$\begin{aligned} I(u) &= \Im\left[1 - \sigma V_t(x) * T(x)\right] \\ &= \Delta(u) - \sigma V_t(u) \cdot T(u) \\ I(u) &= \Delta(u) + 2\sigma V_t(u) \cdot A(u) \cdot E(u) \cdot \sin\chi(u) \end{aligned}$$

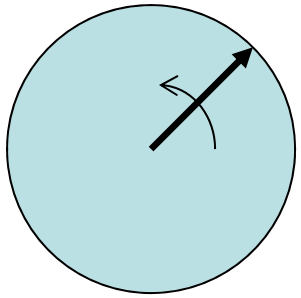
Ideal transfer function for phase object (I)

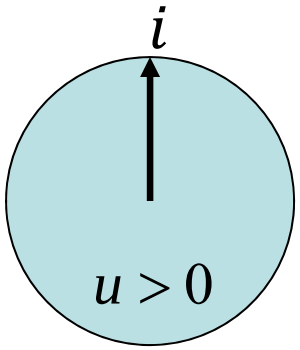
$$\chi(u) = \begin{cases} 0, & |u| < 1/b \\ -\dfrac{\pi}{2}, & |u| \geq 1/b \end{cases}$$

$$H(u) = \mathrm{e}^{-i\chi(u)} = \begin{cases} 1, & |u| < 1/b \\ i, & |u| \geq 1/b \end{cases}$$

$H(u)$





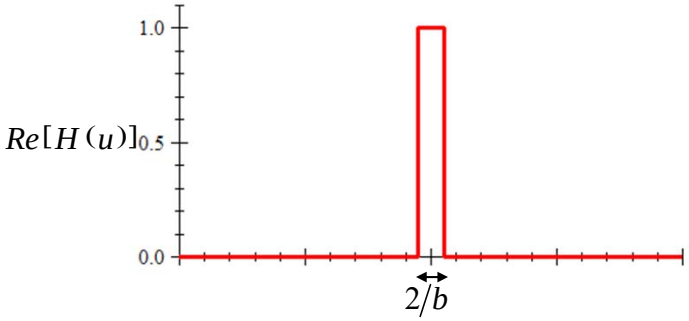


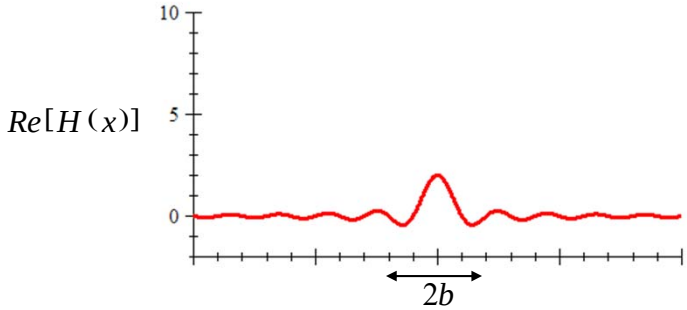
$$T(u) = 2\,\mathrm{Im}\left[H(u)\right] = \begin{cases} 0, & |u| < 1/b \\ 2, & |u| \geq 1/b \end{cases}$$

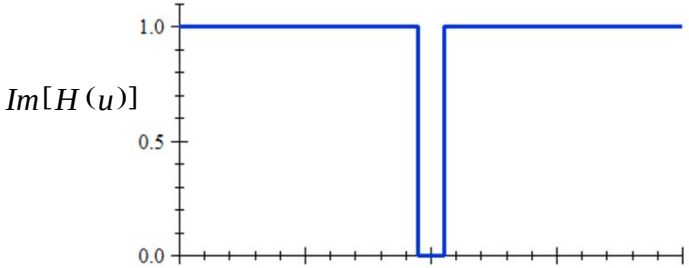
Ideal transfer function for phase object (II)

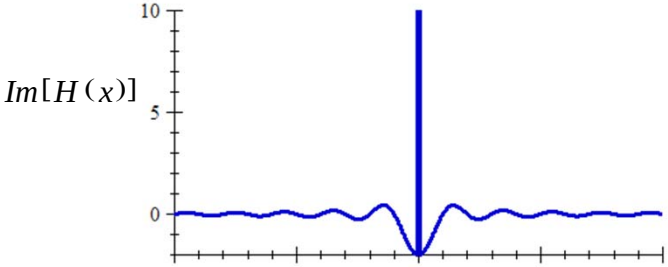
$$H(u) = \mathrm{e}^{-i\chi(u)} = \begin{cases} 1, & |u| < 1/b \\ i, & |u| \geq 1/b \end{cases}$$

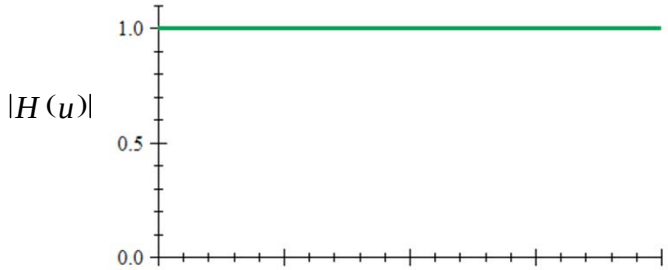
$$H(x) = i\delta(x) + 2(1-i)\cdot \mathrm{sinc}(2\pi x/b)$$

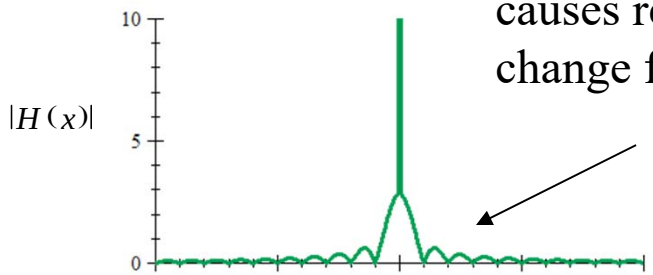












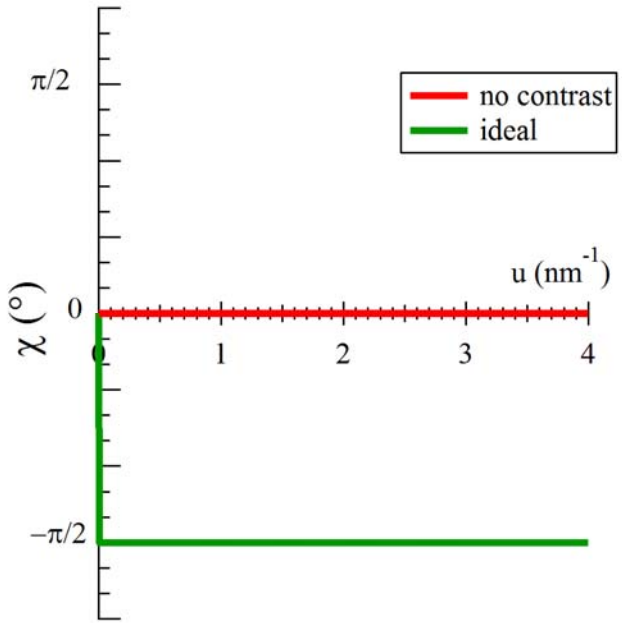
Ideal CTF

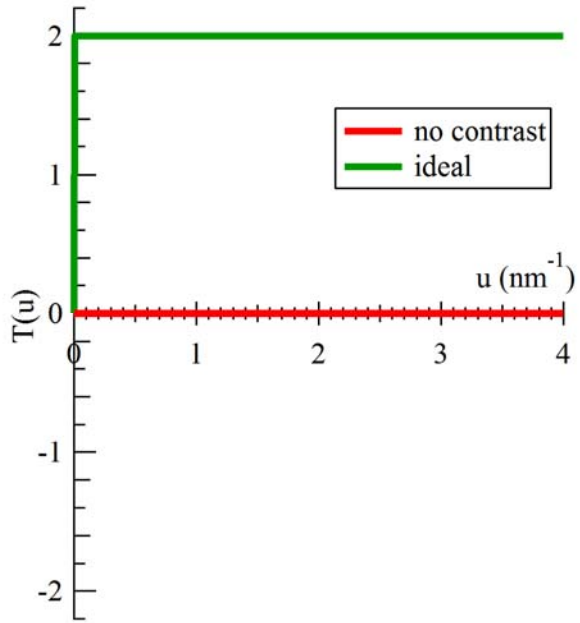
Assume $A(u)E(u)=1$

$$\chi(u)=\begin{cases}0, & u=0\\ -\pi/2, & u>0\end{cases}$$

$$T(u)=-2\sin\chi(u)$$

$$T(u)=\begin{cases}0, & u=0\\ 2, & u>0\end{cases}$$

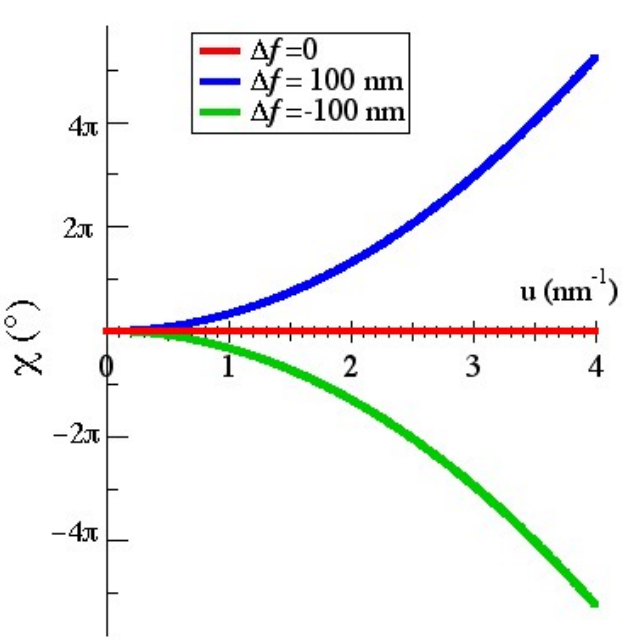


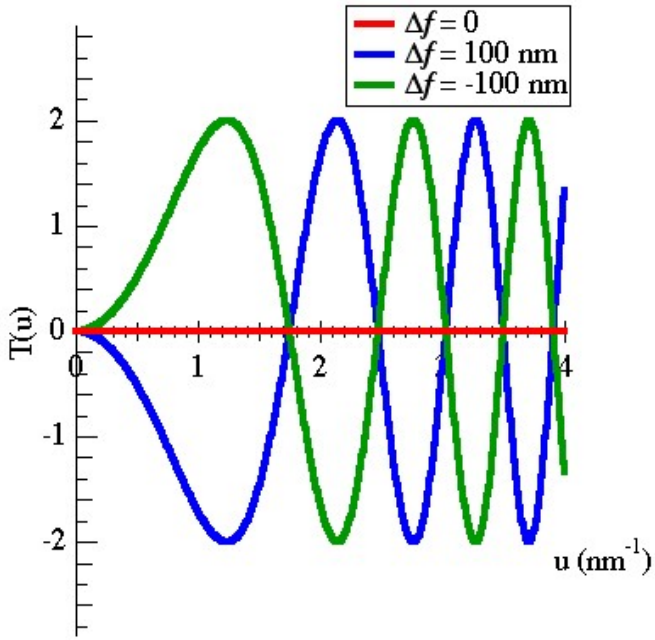


$T(u)>0 \Rightarrow$ positive phase contrast

Real CTF: (case 1: C_s=0)

$$\chi(u) = \pi \cdot \Delta f \cdot \lambda u^2$$



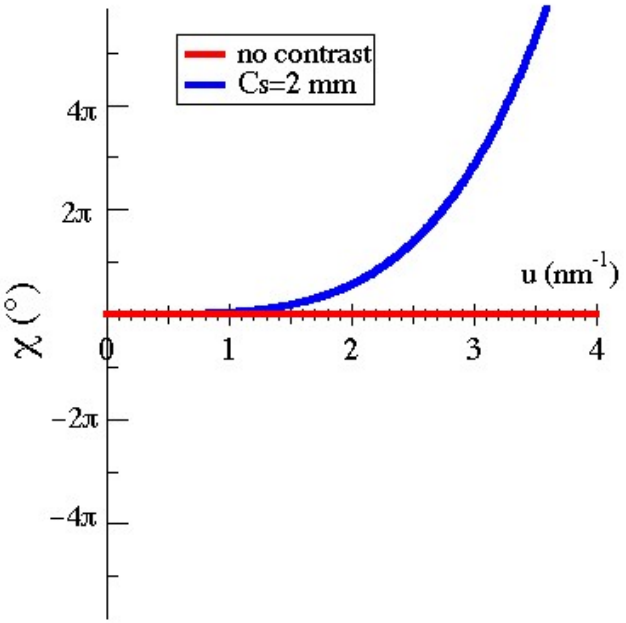


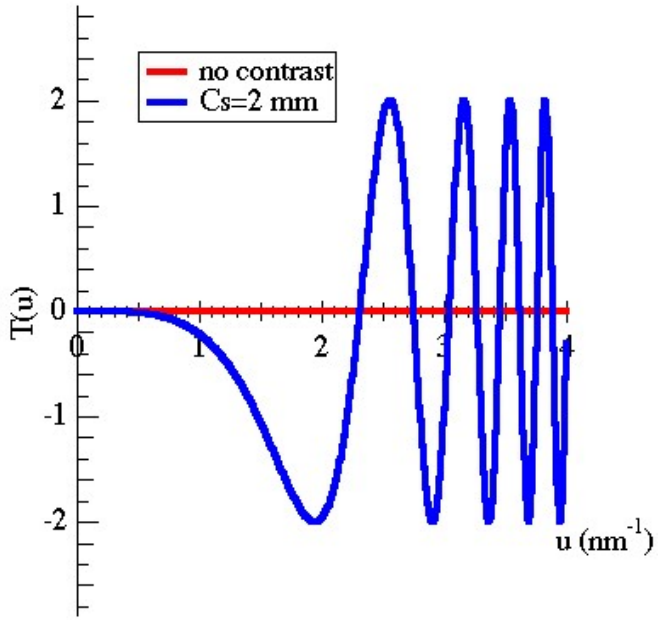
$\Delta f < 0$: underfocusing

$\Delta f > 0$: overfocusing

Real CTF: (case 2: Δf=0)

$$\chi(u) = \frac{1}{2}\pi C_s \lambda^3 u^4$$

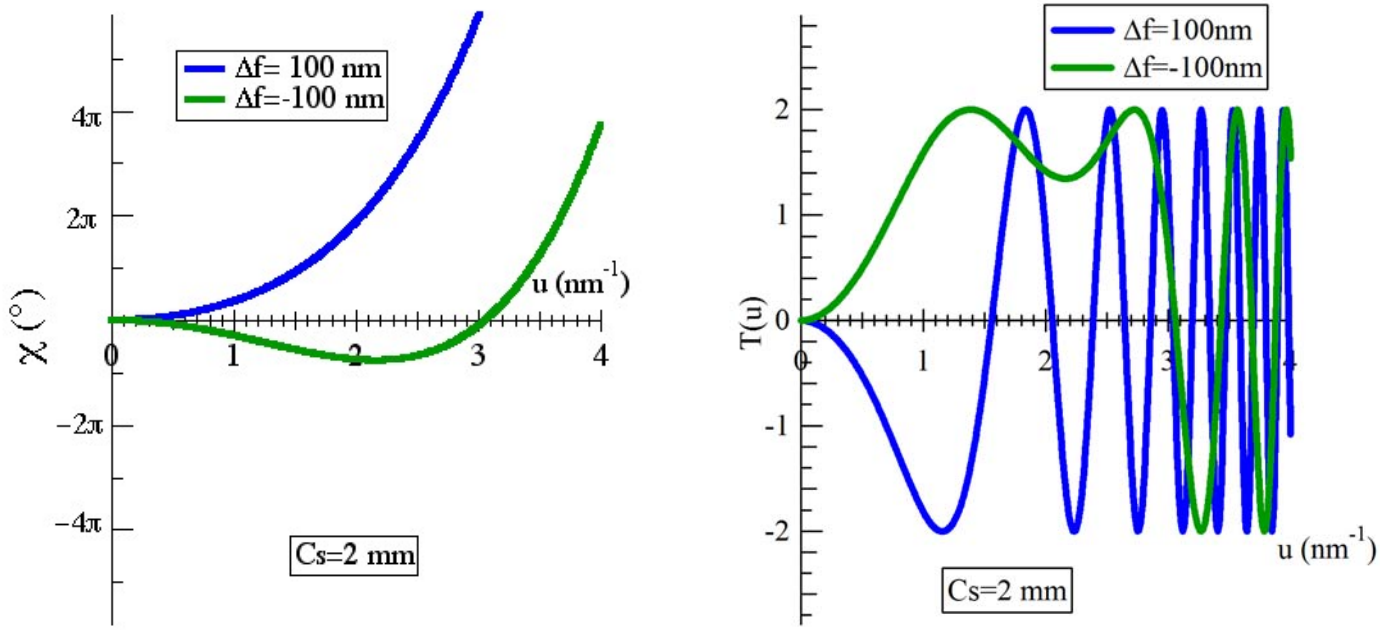




Contributes to negative contrast

Real CTF: (general)

$$\chi(u) = \pi \cdot \Delta f \cdot \lambda u^2 + \frac{1}{2}\pi C_s \lambda^3 u^4$$



Underfocus gives a region with constant phase, positive contrast

Scherzer Defocus

$$\frac{d\chi(u)}{du} = 2\pi \cdot \Delta f \cdot \lambda u + 2\pi C_s \lambda^3 u^3$$

Stationary Phase: $$\left.\frac{d\chi(u)}{du}\right|_{u=u_{\min}} = 0 \Rightarrow \Delta f = \Delta f_{sch} = -C_s \lambda^2 {u_{\min}}^2$$

We need to pick what phase we want stationary:

$$T(u_{\min}) = -2\sin\left[\chi(u_{\min})\right] = \frac{\sqrt{3}}{2} \cdot T_{\max} = \sqrt{3} \qquad \longrightarrow \qquad \sin\left[\chi(u_{\min})\right] = -\frac{\sqrt{3}}{2}$$

$$\chi(u_{\min}) = -\frac{2\pi}{3} = -\frac{1}{2}\pi C_s \lambda^3 {u_{\min}}^4$$

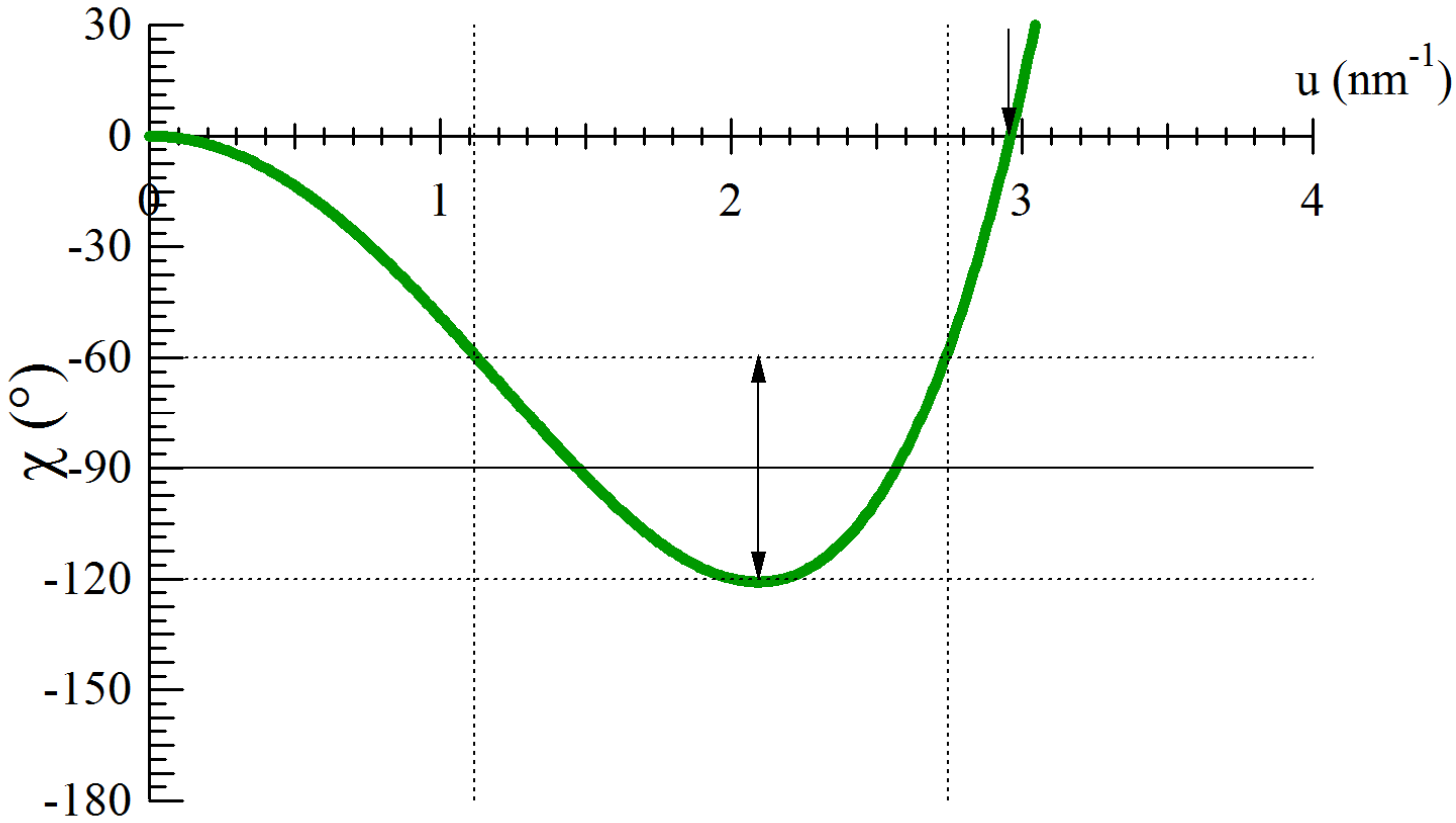
$$u_{\min} = \left(\frac{4}{3C_s\lambda^3}\right)^{1/4} \qquad \Delta f_{sch} = -\left(\frac{4}{3}C_s\lambda\right)^{1/2} \approx -1.2\left(C_s\lambda\right)^{1/2}$$

This choice gives a relatively constant CTF.

Phase At Scherzer defocus

$C_s = 2.0$ mm

$E = 125$ KeV

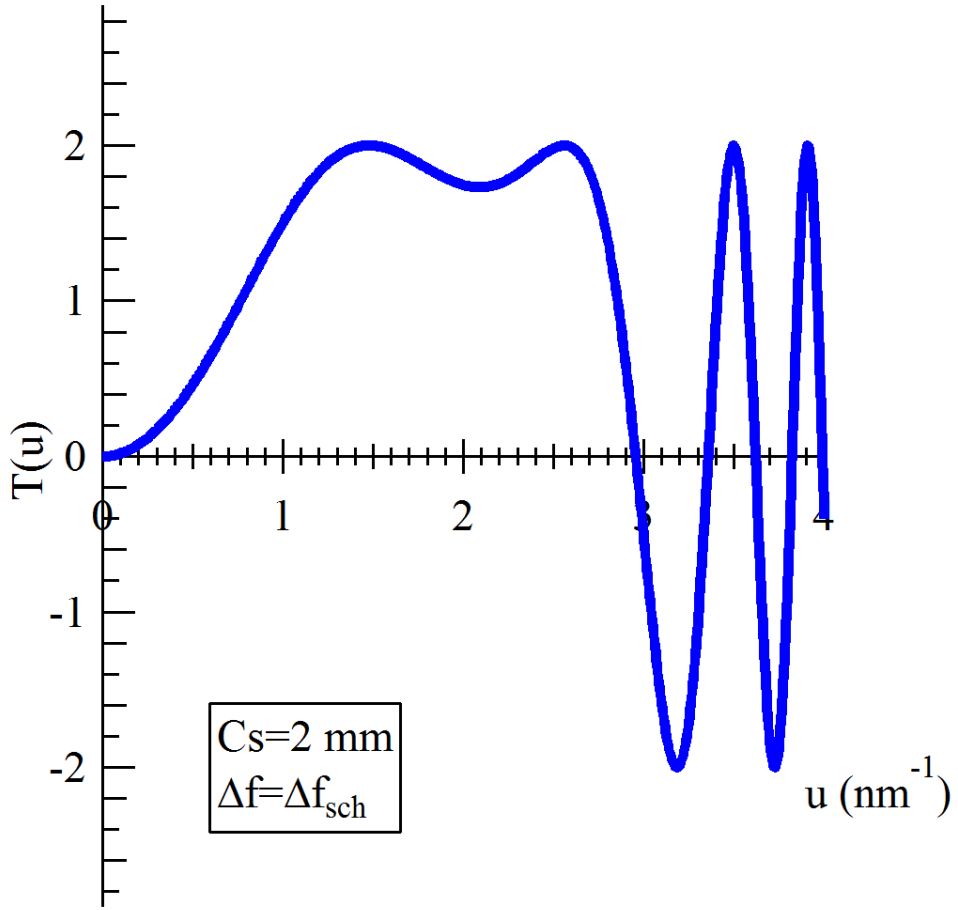


CTF at Scherzer defocus

E=125 keV

C_s=2.0 mm

$\Delta f_{sch} = -93.4$ nm



Resolution at Scherzer defocus

$$\sin\chi\left(u_{sch}\right)=0\Rightarrow\chi\left(u_{sch}\right)=0$$

$$0=\Delta f_{sch}\cdot\lambda u_{sch}{}^{2}+\frac{1}{2}C_s\lambda^{3}u_{sch}{}^{4}$$

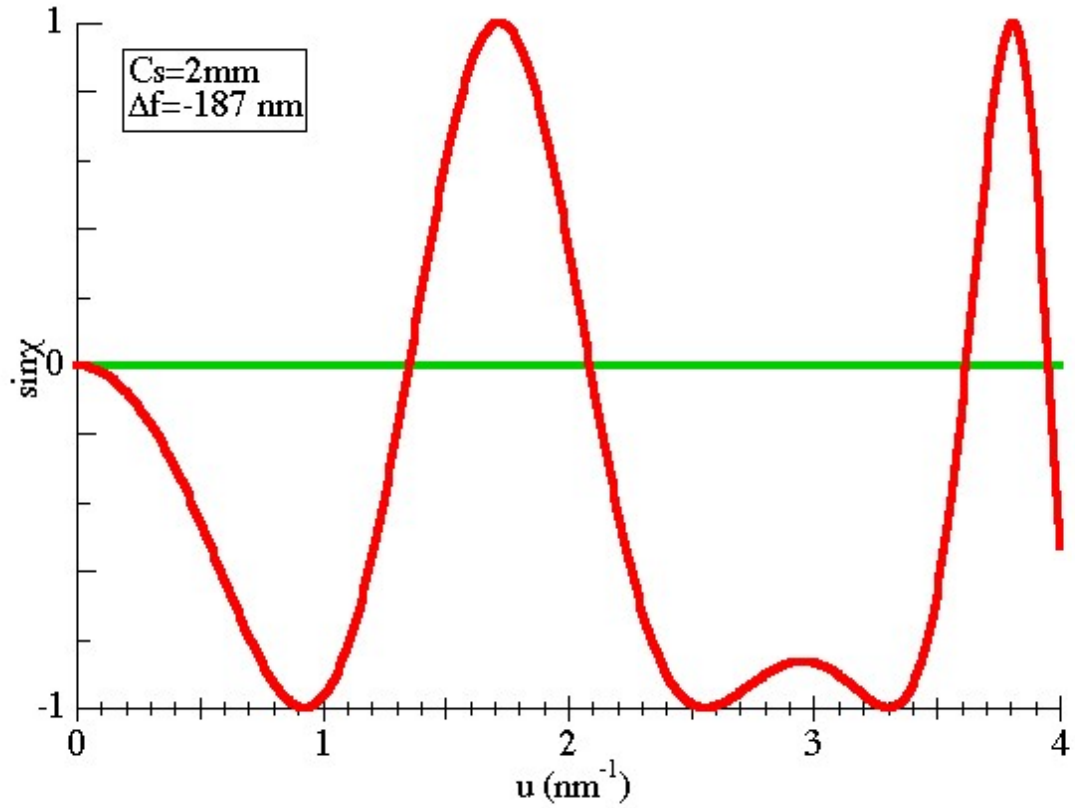
we already know: $\Delta f_{sch}=-\left(\frac{4}{3}C_s\lambda\right)^{1/2}$

$$u_{sch}=2\cdot\left(3C_s\lambda^{3}\right)^{-1/4}\approx 1.51\left(C_s\lambda^{3}\right)^{-1/4}$$

$$d_{sch}=\frac{1}{u_{sch}}\approx 0.66\left(C_s\lambda^{3}\right)^{1/4}$$

Passbands

We could make the phase stationary at higher *u*:



Damping due to temporal incoherence

$$E_c(u) = \exp\left(-\pi^2 \delta^2 \lambda^2 u^4 / 2\right)$$

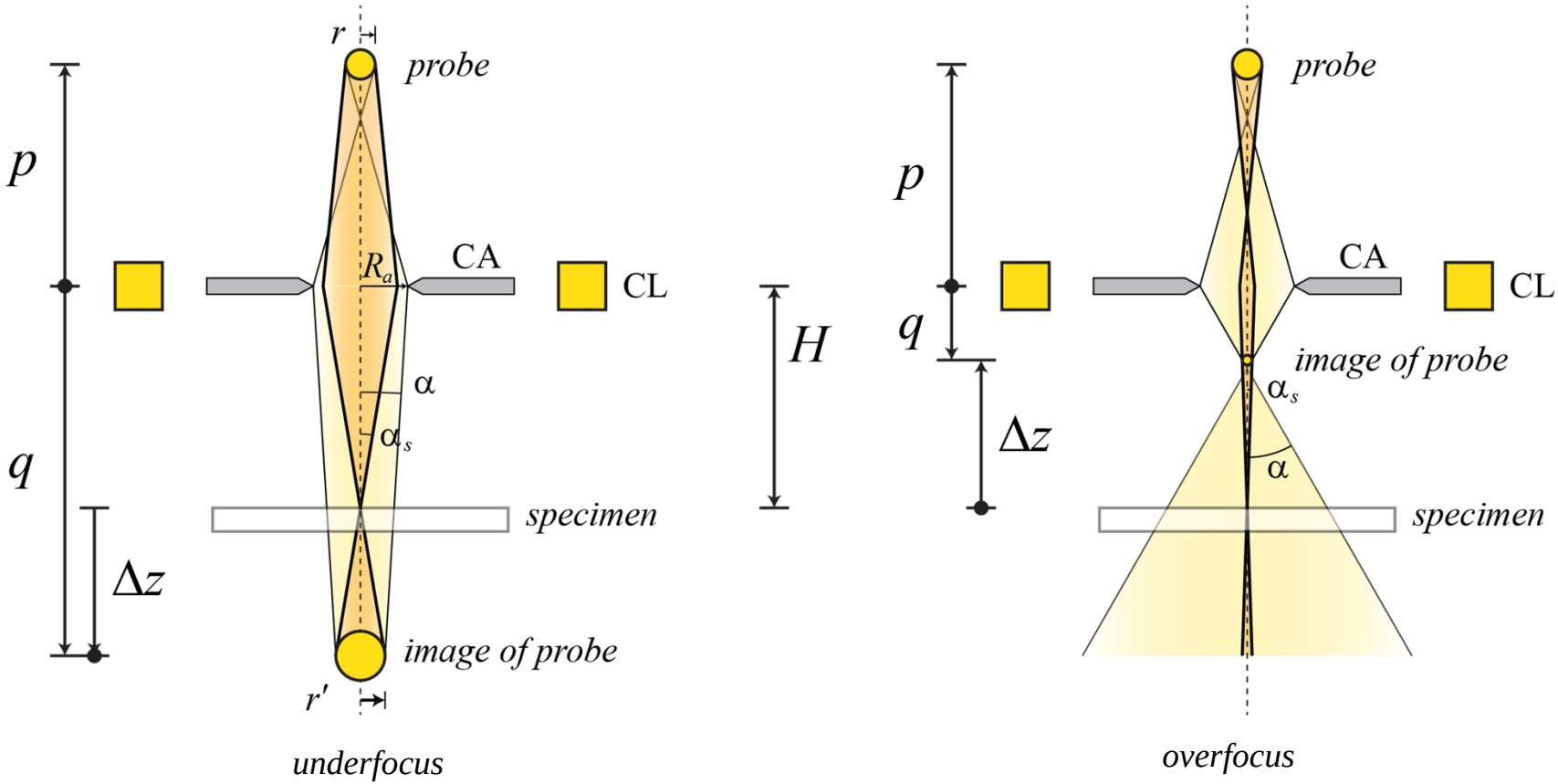
$$\delta = C_c \cdot \sqrt{\left(\frac{\delta E}{E}\right)^2 + \left(\frac{\delta I}{I}\right)^2}$$

chromatic aberration coefficient

$\frac{\delta E}{E}$: Variation in energy

$\frac{\delta I}{I}$: Variation in objective-lens current

Beam convergence: one-lens condenser (I)



A range of illumination angles may be incident on each sample point.
The illumination semiangle α_s is not the same as the beam convergence angle α.

Beam convergence: one-lens condenser (II)

Semi-angle of illumination at a point on the specimen:

$$\alpha_s = \left|\frac{r'}{\Delta z}\right| = \left|\frac{H}{\Delta z} - 1\right| \cdot \alpha_r$$

Illumination semi-angle in lens plane: $\alpha_r = \frac{r}{p}$

Max. illumination angle limited by aperture: $\alpha_s \le \alpha_s^{(\max)}$ $\alpha_s^{(\max)} = \frac{R_a}{H}$

$$\alpha_s = \begin{cases} \left|\frac{r''}{\Delta z}\right| \cdot \alpha_r, & \left|\frac{r''}{\Delta z}\right| \cdot \alpha_r < \alpha_s^{(\max)} \\ \alpha_s^{(\max)}, & \text{otherwise} \end{cases}$$

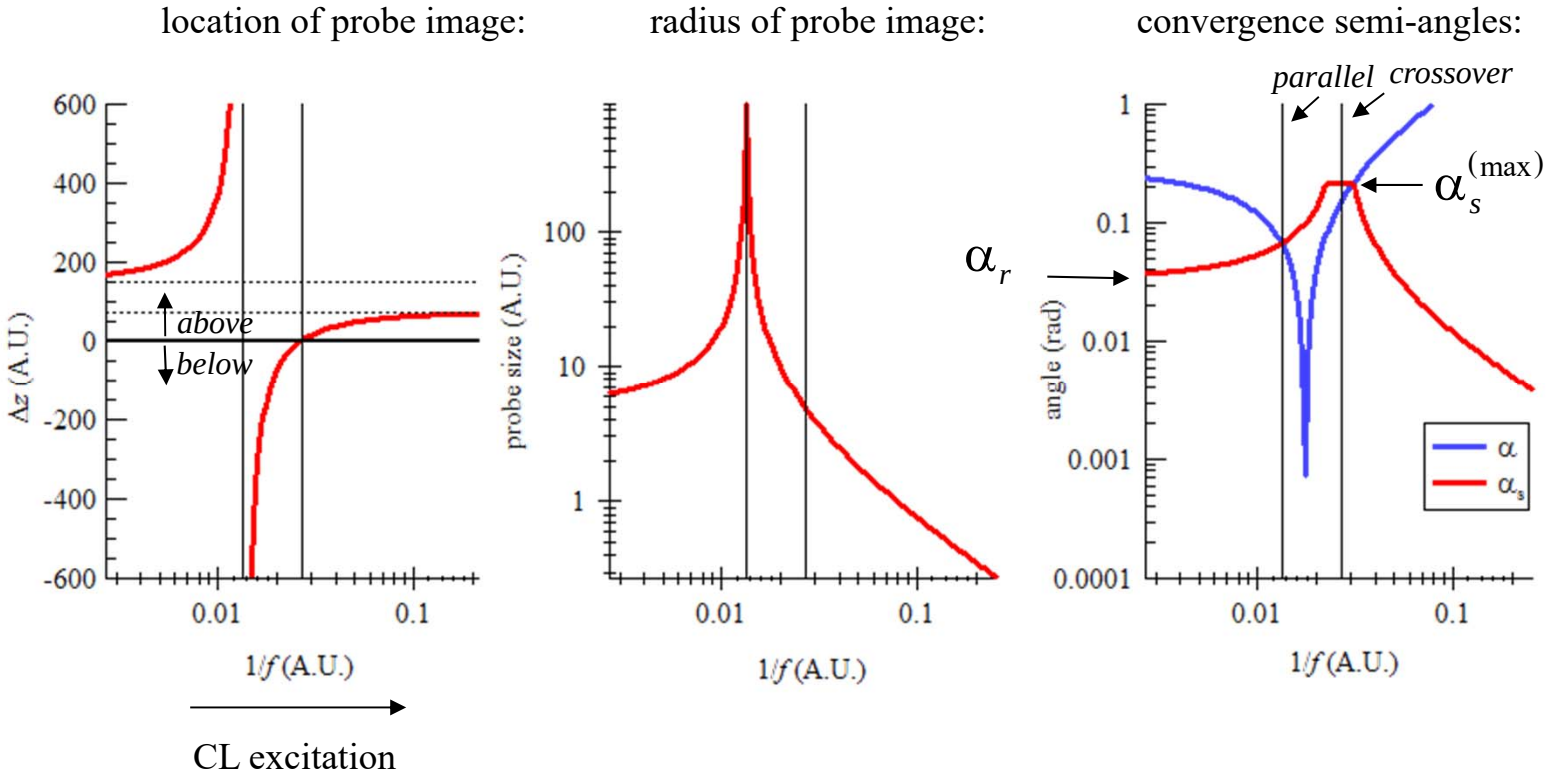
Semi-angle of convergence:

$$\alpha = \left|\frac{R_a}{q_a}\right| = \left|\frac{R_a}{q} - \alpha_r\right|$$

The limiting rays cross the axis at:

$$q_a = \frac{1}{\frac{1}{q} - \left(\frac{r}{R_a}\right) \cdot \frac{1}{p}}$$

Beam convergence: one-lens condenser (III)



Large overfocus is usually better than underfocus.
For underfocus, α_s is never less than α_r .

Beam convergence: two-lens condenser (I)

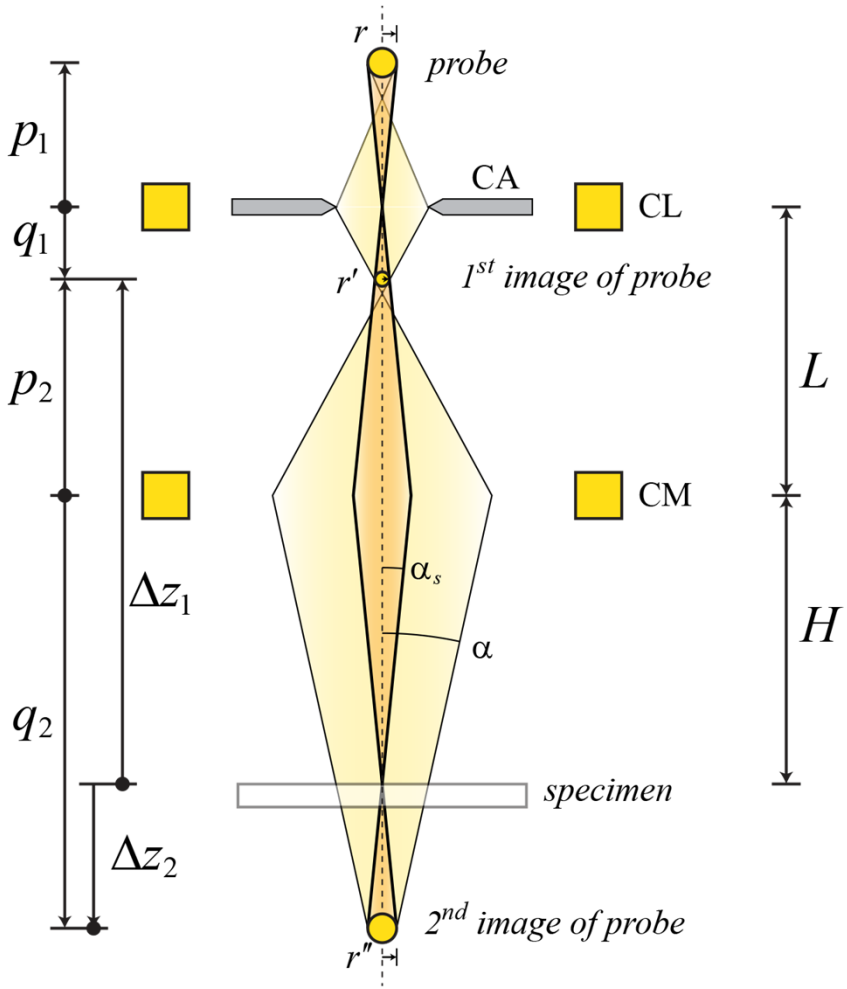
Semi-angle of illumination at a point on the specimen:

$$\alpha_s = \left|\frac{r''}{\Delta z_2}\right| = \frac{\alpha_r}{\left|\left(\frac{L}{q_1}-1\right)\cdot\left(\frac{H}{q_2}-1\right)\right|}$$

$$\alpha_s^{(\max)} = \frac{R_a}{H\cdot L\cdot\left|\left(\frac{1}{L}-\frac{1}{f_2}-\frac{1}{H}\right)\right|}$$

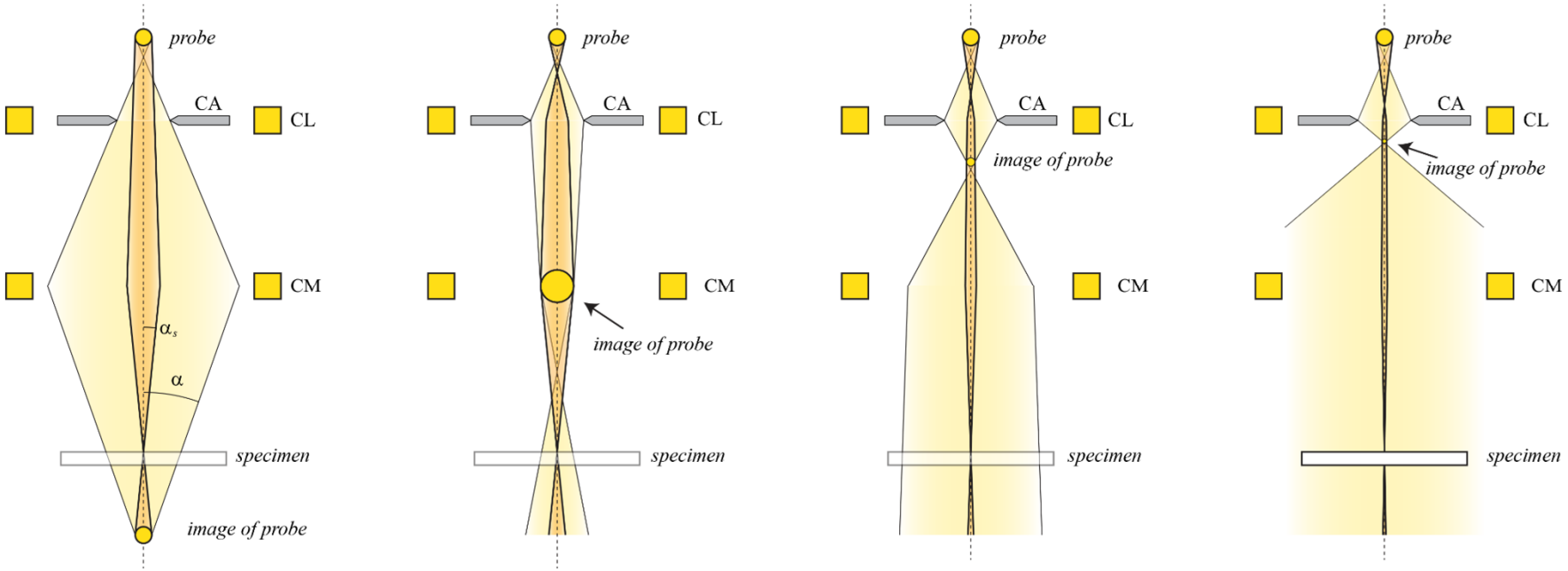
$$\alpha_s = \begin{cases} \left|\frac{r''}{\Delta z_2}\right|, & \left|\frac{r''}{\Delta z_2}\right|\cdot\alpha_r < \alpha_s^{(\max)} \\ \alpha_s^{(\max)}, & \text{otherwise} \end{cases}$$

$\alpha_r = r/p_1$



Beam convergence: two-lens condenser (II)

Two-lens condenser system:



underfocus (CL off)

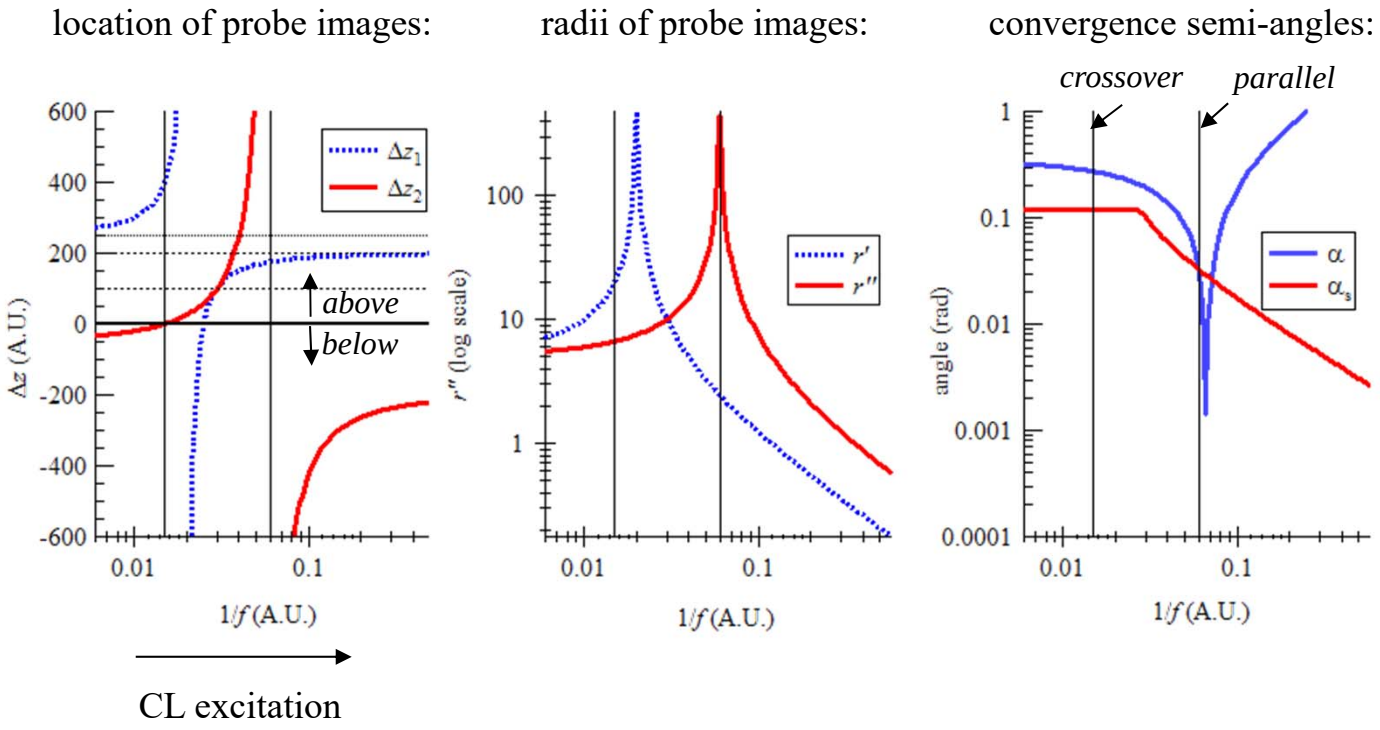
CL slightly overfocused

nearly parallel

nearly optimal (CL highly overfocused)

Allows small, parallel beam on sample.
Parallel beam formed when probe image formed by CL is at front focal point of CM.
Highly overfocused CL gives small illumination semi-angle on a sample point.

Beam convergence: two-lens condenser (III)



Beyond crossover, α_s decreases with increasing CL3 strength ("Brightness").
Parallel beam formed with CL3 overfocus to specific value; good for diffraction.
Probe size increases with increasing CL3 strength below crossover.

Damping due to beam convergence (I)

Assuming a parallel beam: $G_0(u) = F_0(u) \cdot H_0(u)$

Intensity: $I_0(u) = \int_{u'=-\infty}^{\infty} du' \cdot G_0^{*}(u-u') \cdot G_0(u')$

For a non-parallel source, we have to integrate over *u*:

Gaussian spot profile: $S(u) = \frac{1}{\sqrt{\pi} k \alpha_s} \cdot \mathrm{e}^{-\left(\frac{u}{k\alpha_s}\right)^2}$

There are two distinct ways to combine the incidence angles: coherently and incoherently

coherent: $G(u) = \int_{u'=-\infty}^{\infty} du' \cdot G_0(u-u') \cdot S(u')$

incoherent: $I(u) = \int_{u'=-\infty}^{\infty} du' \cdot I_0(u-u') \cdot S(u')$

Damping due to beam convergence (II)

Assume only effect is a phase shift : $G_0(u) = F(u) \cdot \mathrm{e}^{-i\chi(u)}$

Let's assume the different incidence angles combine coherently.

$$G(u) = \frac{1}{\sqrt{\pi} k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot F(u-u') \cdot \mathrm{e}^{-i\chi(u-u')} \cdot \mathrm{e}^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

If the input is a delta function, the output is the transfer function:

$$H(x) = S[\delta(x)]$$

$$F(x) = \delta(x) \rightarrow F(u) = 1$$

$$H(u) = \frac{1}{\sqrt{\pi} k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot \mathrm{e}^{-i\chi(u-u')} \cdot \mathrm{e}^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

Damping due to spatial incoherence (II)

Assume $k\alpha_s$ is small.

$$\chi(u-u') \approx \chi(u) - u' \cdot \left.\frac{\partial \chi(u'')}{\partial u''}\right|_{u''=u} = \chi(u) - u' \cdot C(u)$$

Do the integral: $$H(u) \approx e^{-i\chi(u)} \cdot \frac{1}{\sqrt{\pi} k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot e^{-iC(u)\cdot u'} \cdot e^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

$$= e^{-i\chi(u)} \cdot \frac{1}{\sqrt{\pi} k\alpha_s} \cdot \int_{u'=-\infty}^{\infty} du' \cdot \cos\left[C(u)\cdot u'\right] \cdot e^{-\left(\frac{u'}{k\alpha_s}\right)^2}$$

$$H(u) = e^{-i\chi(u)} \cdot \exp\left[-\left(\frac{k\alpha_s}{2} \cdot C(u)\right)^2\right] = e^{-i\chi(u)} \cdot E_s(u)$$

Spatial incoherence is represented by a damping function:

$$E_s(u) = \exp\left[-\left(\frac{k\alpha_s}{2} \cdot \frac{\partial \chi}{\partial u}\right)^2\right]$$

Combine incoherence effects

We can combine temporal and spatial incoherence functions:

$$E(u) = E_c(u) \cdot E_s(u) \cdots$$

Finding path-length difference for an electron

$\psi(\mathbf{r}) = \mathrm{e}^{i\phi(\mathbf{r})}$ //electron wave function

$\hat{\mathbf{p}} = -ih\nabla$

$\hat{\mathbf{p}}\psi = \left[h \cdot \nabla\phi(\mathbf{r})\right] \cdot \psi(\mathbf{r})$ //momentum operator

$\nabla\phi(\mathbf{r}) = \mathbf{p}/h$

$\delta\phi(\mathbf{r}) = \dfrac{2\pi}{h} \cdot \int_{\mathbf{r}=\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{p} \cdot d\mathbf{r} = 2\pi \cdot \delta n(\mathbf{r})$ //momentum proportional to gradient of phase // phase change

$\delta n(\mathbf{r}) = \dfrac{1}{h} \cdot \int_{\mathbf{r}=\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{p} \cdot d\mathbf{r}$ //change in number of wave fronts

$d\mathbf{r} = d\rho \cdot \hat{\boldsymbol{\rho}} + \rho \cdot d\phi \cdot \hat{\boldsymbol{\phi}} + dz \cdot \hat{\mathbf{z}}$

$\mathbf{p} \cdot d\mathbf{r} = m \cdot \left(\dot{\rho} \cdot d\rho + \dot{z} \cdot dz\right)$ //momentum in uniform-field lens

$\dot{\rho} = d\rho/dt \qquad \dot{z} = dz/dt$

$\mathbf{p} \cdot d\mathbf{r} = m \cdot \left(\dot{\rho}^2 + \dot{z}^2\right) \cdot dt$ //needed to find phase change

Uniform-Field Electron Lens (II)

$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$ //canonical momentum

$\mathbf{B} = \nabla \times \mathbf{A} = B_\rho \hat{\boldsymbol{\rho}} + B_z \hat{\mathbf{z}}$ //magnetic field in axially symmetric lens

$B_\rho = -\frac{\rho B_0}{2} \cdot \delta(z+a) + \frac{\rho B_0}{2} \delta(z-a)$ //uniform-field model

$B_z = B_0 \cdot [u(z+a) - u(z-a)]$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \cdot \hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \cdot \hat{\boldsymbol{\phi}} + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \cdot \hat{\mathbf{z}}$$

$\mathbf{A} = \frac{\rho B_0}{2} \cdot [u(z+a) - u(z-a)] \hat{\boldsymbol{\phi}} = \frac{m\omega_L \rho}{e} \cdot [u(z+a) - u(z-a)] \hat{\boldsymbol{\phi}}$ //magnetic vector potential

Uniform-Field Electron Lens (II)

$\mathbf{v} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$ //velocity

Assume: $\dot{\phi} = 0 \ (z < -a)$

Inside the lens: $\dot{\phi} = \omega_L$

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} = m\mathbf{v} - e\mathbf{A}$$
$$= m\left(\dot{\rho}\hat{\boldsymbol{\rho}} + \rho\omega_L\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}\right) - m\omega_L\rho\hat{\boldsymbol{\phi}}$$
$$\mathbf{p} = m\left(\dot{\rho}\hat{\boldsymbol{\rho}} + \dot{z}\hat{\mathbf{z}}\right)$$

$$\rho = \rho_0 \cdot \cos\left(\omega_L \cdot t + \theta\right)$$

$$\dot{\rho} = -\omega_L \cdot \rho_0 \cdot \sin\left(\omega_L \cdot t + \theta\right)$$

$$\mathbf{p} \cdot d\mathbf{r} = m \cdot v_z'^2 \cdot \left\{1 + \left(\frac{\omega_L \cdot \rho_0}{v_z'}\right)^2 \cdot \sin^2\left(\omega_L \cdot t + \theta\right)\right\} \cdot dt$$ //differential change

Uniform-Field Electron Lens (III)

$$m \cdot v'_{\mathbf{z}} = h/\lambda'_z \qquad \lambda'_z = \lambda_z \Big/ \sqrt{1-u^2}$$

$$\delta n(t) \cdot \lambda'_z = \frac{1}{m \cdot v'_{\mathbf{z}}} \cdot \int_{\mathbf{r}=\mathbf{r}(0)}^{\mathbf{r}(t)} \mathbf{p} \cdot d\mathbf{r}$$

$$\delta n(t) \cdot \lambda'_z = v'_z \int_{t'=0}^{t} \left\{ 1 + \left(\frac{\omega_L \cdot \rho_0}{v'_z} \right)^2 \cdot \sin^2\left(\omega_L \cdot t' + \theta\right) \right\} \cdot dt' \qquad \text{//differential path length}$$

Integrate:

$$\delta n(t) \cdot \lambda'_z = v'_z \cdot \left\{ t + \left(\frac{\omega_L \cdot \rho_0}{v'_z} \right)^2 \cdot \left[\frac{1}{2} \left\{ t - \frac{1}{2\omega_L} \left(\sin\left(2\left[\omega_L \cdot t + \theta\right]\right) - \sin\left(2\theta\right) \right) \right\} \right] \right\}$$

Combine terms:

$$\delta n(t) \cdot \lambda'_z = v'_z \cdot \left\{ \left[1 + \frac{1}{2} \left(\frac{\omega_L \cdot \rho_0}{v'_z} \right)^2 \right] \cdot t - \frac{1}{4\omega_L} \cdot \left(\frac{\omega_L \cdot \rho_0}{v'_z} \right)^2 \cdot \left(\sin\left(2\left[\omega_L \cdot t + \theta\right]\right) - \sin\left(2\theta\right) \right) \right\}$$

Simplify

$$\delta n(t) \cdot \lambda_z = v_z \cdot \left\{ \left(1 - \frac{1}{2} u^2 \right) \cdot t - \frac{1}{4\omega_L} \cdot u^2 \cdot \left(\sin\left(2\left[\omega_L \cdot t + \theta\right]\right) - \sin\left(2\theta\right) \right) \right\}$$

Uniform-Field Electron Lens (IV)

Assume: $u^2 \ll 1$ $\qquad 1-\frac{1}{2}u^2 \approx \frac{1}{\sqrt{1-u^2}}$

$$\delta n(t)\cdot\lambda_z \approx v_z \cdot\left\{\frac{t}{\sqrt{1-u^2}}-\frac{1}{4\omega_L}\cdot u^2\cdot\left(\sin\left(2\left[\omega_L\cdot t+\theta\right]\right)-\sin\left(2\theta\right)\right)\right\}$$

Switch to z: $z+a=v_z'\cdot t$ $\qquad k=\omega_L/v_z$ $\qquad k'=\omega_L/v_z'$

$$\delta n(t)\cdot\lambda_z = z+a-\frac{u^2}{4k}\cdot\left\{\sin\left(2\left[k'\left(z+a\right)+\theta\right]\right)-\sin\left(2\theta\right)\right\}$$

Ignore spherical aberration: $k'\to k$

$$\delta n(z)\cdot\lambda_z = z+a-\frac{u^2}{4k}\left(\sin\left\{2\left[k\left(z+a\right)+\theta\right]\right\}-\sin\left(2\theta\right)\right)$$

Refer to lens dimension *a*:

$$\delta n(z)\cdot\frac{\lambda_z}{a} = \frac{z}{a}+1-\frac{u^2}{4ka}\left(\sin\left\{2\left[ka\cdot\left(\frac{z}{a}+1\right)+\theta\right]\right\}-\sin\left(2\theta\right)\right)$$

Uniform-Field Electron Lens (V)

The electron wavelength varies along its trajectory. The momentum is

$$p(z) = m \cdot \sqrt{v^2 - v_\phi{}^2(z)}$$

$$mv = h/\lambda \qquad v_\phi = -\omega_L \cdot \rho$$

$$\frac{1}{\lambda'(z)} = \frac{1}{\lambda} \cdot \sqrt{1 - k^2 \cdot \rho^2(z)}$$

The wavelength increases at larger radius, along the wave fronts to combine in phase at the crossovers.

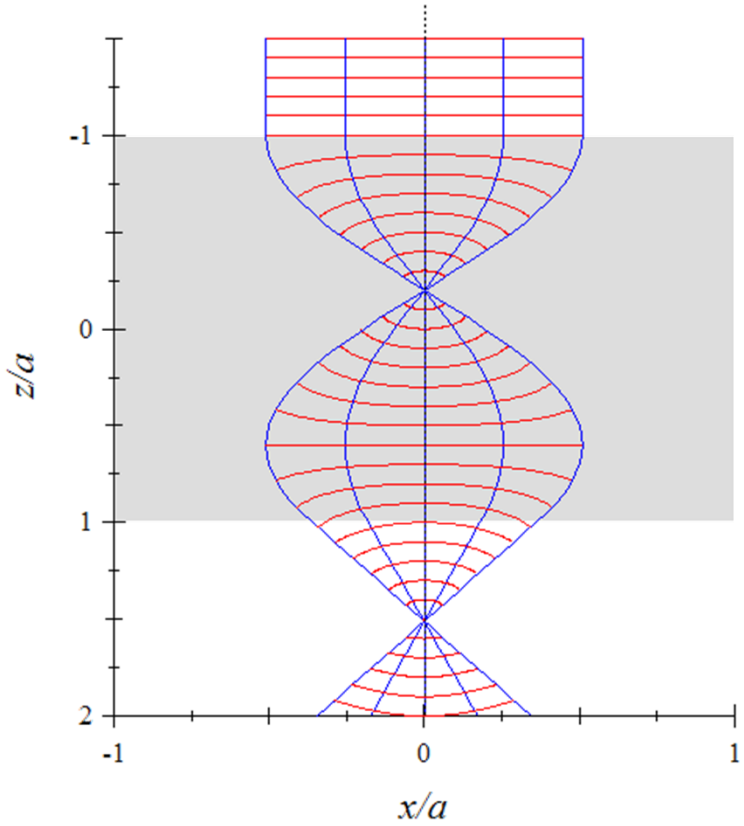


Image of periodic specimen (I)

Object function:

$$F(x) = \sum_g F_g \cdot \mathrm{e}^{2\pi i g x}$$

$$\begin{aligned} F(u) &= \Im\{F(x)\} \\ &= \sum_g F_g \cdot \{\Im[\mathrm{e}^{2\pi i g x}]\} \\ F(u) &= \sum_g F_g \cdot \Delta(u-g) \end{aligned}$$

Transfer function (no attenuation):

$$H(u) = A(u) \cdot \mathrm{e}^{-i\chi(u)}$$

$$G(u) = \left[\sum_g F_g \cdot \Delta(u-g)\right] \cdot A(u) \cdot \mathrm{e}^{-i\chi(u)}$$

Image function:

$$\begin{aligned} G(x) &= \lim_{K\to\infty}\left[\int_{u=-K}^{K} G(u)\cdot \mathrm{e}^{2\pi i u x}\cdot du\right] \\ &= \lim_{K\to\infty}\left\{\int_{u=-K}^{K} \sum_g F_g \cdot \Delta(u+g)\cdot A(u)\cdot \mathrm{e}^{-i\chi(u)}\cdot \mathrm{e}^{2\pi i u x}\cdot du\right\} \\ G(x) &= \sum_g F_g \cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\cdot \mathrm{e}^{2\pi i g x} \end{aligned}$$

Image of periodic specimen (II)

FT of image function:

$$
\begin{aligned}
G(u) &= \lim_{L\to\infty}\left[\int_{x=-L}^{L} G(x)\cdot \mathrm{e}^{-2\pi iux}\cdot dx\right] \\
&= \lim_{L\to\infty}\int_{x=-L}^{L}\left\{\sum_g \left[F_g\cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\cdot \mathrm{e}^{2\pi igx}\right]\cdot \mathrm{e}^{-2\pi iux}\cdot dx\right\} \\
&= \sum_g\left\{\lim_{L\to\infty}\int_{x=-L}^{L}\left[F_g\cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\cdot \mathrm{e}^{2\pi igx}\right]\cdot \mathrm{e}^{-2\pi iux}\cdot dx\right\} \\
&= \sum_g F_g\cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\left\{\lim_{L\to\infty}\int_{x=-L}^{L}\mathrm{e}^{-2\pi i(g-u)x}\cdot dx\right\} \\
&= \sum_g F_g\cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}\Delta(u-g) \\
G(u) &= \sum_g G_g\cdot\Delta(u-g)
\end{aligned}
$$

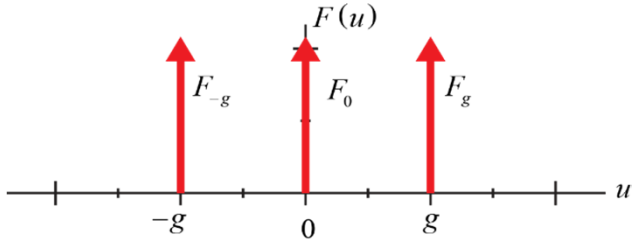
$$G_g = F_g\cdot A(g)\cdot \mathrm{e}^{-i\chi(g)}$$

$$G(x) = \sum_g G_g\cdot \mathrm{e}^{2\pi igx}$$

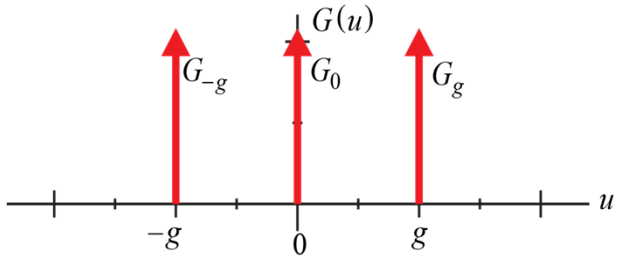
Image of periodic specimen (III)

Three-beam case:

$$F(u) = F_{-g} \cdot \Delta(u+g) + F_0 \cdot \Delta(u) + F_g \cdot \Delta(u-g)$$



$$G(u) = G_{-g} \cdot \Delta(u+g) + G_0 \cdot \Delta(u) + G_g \cdot \Delta(u-g)$$



	$-g$	0	g
$F(u)$	F_{-g}	F_0	F_g
$A(u)$	$A(-g)$	$A(0)$	$A(g)$
$e^{-i\chi(u)}$	$e^{-i\chi(-g)}$	$e^{-i\chi(0)}$	$e^{-i\chi(g)}$
$G(u)$	$F_{-g} \cdot A(-g) \cdot e^{-i\chi(-g)}$	$F_0 \cdot A(0) \cdot e^{-i\chi(0)}$	$F_g \cdot A(g) \cdot e^{-i\chi(g)}$

Example

Given: $F(x) = 1 + iB \cdot \sin(2\pi g x)$ $\qquad A(u) = 1$ $\qquad \chi(u) = \begin{cases} 0, & |u| \le g/2 \\ -\pi/2, & g/2 < |u| \end{cases}$

Write: $F(x) = -\frac{B}{2} e^{-2\pi i g x} + 1 + \frac{B}{2} e^{2\pi i g x} = F_{-g} e^{-2\pi i g x} + F_0 + F_g e^{2\pi i g x}$

	$-g$	0	g
$F(u)$	$-B/2$	1	$B/2$
$A(u)$	1	1	1
$\chi(u)$	$-\pi/2$	0	$-\pi/2$
$e^{-i\chi(u)}$	i	1	i
$G(u)$	$-iB/2$	1	$iB/2$

$\Rightarrow$ $G(x) = -\frac{iB}{2} e^{-2\pi i g x} + 1 + \frac{iB}{2} e^{2\pi i g x} = 1 - B \sin(2\pi g x)$