23. High-Resolution Imaging

Problems

23.1. The object wave function below a sample is:

$$F(x) = 1 + iB \cdot \cos(2\pi g x).$$

The microscope transfer function in reciprocal space is given by $H(u) = A(u) \cdot e^{-i\chi(u)}$.

Find the image wave function $G(x)$ in direct space for the following cases:

a) $A(u) = \begin{cases} 1, & |u| \le g/2 \\ 0, & g/2 < |u| \end{cases}$, $\chi(u) = \begin{cases} 0, & |u| \le g/2 \\ \pi/2, & g/2 < |u| \end{cases}$

b) $A(u) = \begin{cases} 1, & |u| \le 3g/2 \\ 0, & 3g/2 < |u| \end{cases}$, $\chi(u) = \begin{cases} 0, & |u| \le g/2 \\ \pi/2, & g/2 < |u| \end{cases}$

c) $A(u) = \begin{cases} 1, & |u - g/2| \le g \\ 0, & g < |u - g/2| \end{cases}$, $\chi(u) = \begin{cases} 0, & |u| \le g/2 \\ \pi, & g/2 < |u| \end{cases}$

23.1.

$G(x) = F(x) * H(x)$ //image function

$H(u) = A(u) \cdot e^{-i\chi(u)}$ //transfer function

$$G(u) = F(u) \cdot H(u) = F(u) \cdot A(u) \cdot e^{-i\chi(u)}$$

Assume object function is periodic:

$$F(x) = \sum_g F_g \cdot e^{2\pi i g x}$$

$$\begin{aligned} F(u) &= \Im\{F(x)\} \\ &= \Im\left\{\sum_g F_g \cdot e^{2\pi i g x}\right\} \\ &= \int_x \left(\sum_g F_g \cdot e^{2\pi i g x}\right) \cdot e^{-2\pi i u x} \cdot dx \\ &= \sum_g \left[\int_x F_g^{-2\pi i (u-g) x} \cdot dx\right] \\ &= \sum_g F_g \cdot \left[\int_x {}^{-2\pi i (u-g) x} \cdot dx\right] \end{aligned}$$

$$F(u) = \sum_g F_g \cdot \Delta(u - g)$$

General form for the image function (also periodic):

$$G(u) = \left[\sum_g F_g \cdot \Delta(u - g)\right] \cdot A(u) \cdot e^{-i\chi(u)}$$

$$G(x) = \lim_{K \to \infty} \left[\int_{u=-K}^{K} G(u) \cdot e^{2\pi iux} \cdot du \right]$$

$$= \lim_{K \to \infty} \left\{ \int_{u=-K}^{K} \sum_g F_g \cdot \Delta(u+g) \cdot A(u) \cdot e^{-i\chi(u)} \cdot e^{2\pi iux} \cdot du \right\}$$

$$G(x) = \sum_g F_g \cdot A(g) \cdot e^{-i\chi(g)} \cdot e^{2\pi igx}$$

Find the Fourier coeffs. of the image function:

$$G(u) = \lim_{L \to \infty} \left[\int_{x=-L}^{L} G(x) \cdot e^{-2\pi iux} \cdot dx \right]$$

$$= \lim_{L \to \infty} \int_{x=-L}^{L} \left\{ \sum_g \left[F_g \cdot A(g) \cdot e^{-i\chi(g)} \cdot e^{2\pi igx} \right] \cdot e^{-2\pi iux} \cdot dx \right\}$$

$$= \sum_g \left\{ \lim_{L \to \infty} \int_{x=-L}^{L} \left[F_g \cdot A(g) \cdot e^{-i\chi(g)} \cdot e^{2\pi igx} \right] \cdot e^{-2\pi iux} \cdot dx \right\}$$

$$= \sum_g F_g \cdot A(g) \cdot e^{-i\chi(g)} \left\{ \lim_{L \to \infty} \int_{x=-L}^{L} e^{-2\pi i(g-u)x} \cdot dx \right\}$$

$$= \sum_g F_g \cdot A(g) \cdot e^{-i\chi(g)} \Delta(u-g)$$

$$G(u) = \sum_g G_g \cdot \Delta(u-g)$$

So,

$$G_g = F_g \cdot A(g) \cdot e^{-i\chi(g)}$$

$$G(x) = \sum_g G_g \cdot e^{2\pi igx}$$

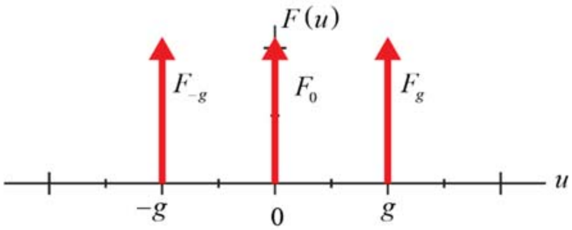
General form for a typical problem:

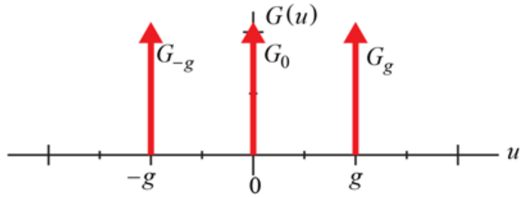
$$F(u) = F_{-g} \cdot \Delta(u+g) + F_0 \cdot \Delta(u) + F_g \cdot \Delta(u-g)$$

$$G(u) = G_{-g} \cdot \Delta(u+g) + G_0 \cdot \Delta(u) + G_g \cdot \Delta(u-g)$$

Make a table:

	-g	0	g
$F(u)$	F_{-g}	F_0	F_g
$A(u)$	$A(-g)$	$A(0)$	$A(g)$
$e^{-i\chi(u)}$	$e^{-i\chi(-g)}$	$e^{-i\chi(0)}$	$e^{-i\chi(g)}$
$G(u)$	$F_{-g} \cdot A(-g) \cdot e^{-i\chi(-g)}$	$F_0 \cdot A(0) \cdot e^{-i\chi(0)}$	$F_g \cdot A(g) \cdot e^{-i\chi(g)}$





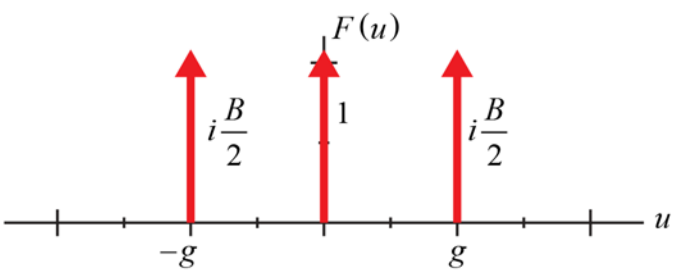
In this case:

$$F(x) = 1 + iB \cdot \cos(2\pi g x)$$

$$= 1 + iB \cdot \left(\frac{e^{2\pi i g x} + e^{-2\pi i g x}}{2} \right)$$

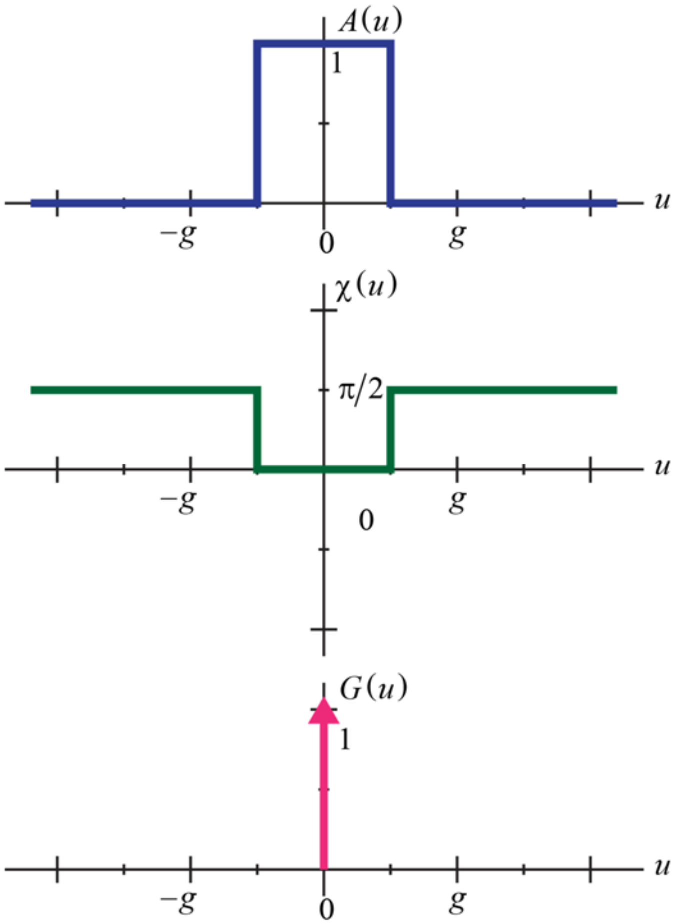
$$F(x) = i\frac{B}{2} e^{-2\pi i g x} + 1 + i\frac{B}{2} e^{2\pi i g x}$$

	-g	0	g
$F(u)$	$i\frac{B}{2}$	1	$i\frac{B}{2}$
$A(u)$	$A(-g)$	$A(0)$	$A(g)$
$e^{-i\chi(u)}$	$e^{-i\chi(-g)}$	$e^{-i\chi(0)}$	$e^{-i\chi(g)}$
$G(u)$	$i\frac{B}{2} \cdot A(-g) \cdot e^{-i\chi(-g)}$	$A(0) \cdot e^{-i\chi(0)}$	$i\frac{B}{2} \cdot A(g) \cdot e^{-i\chi(g)}$



a) $A(u)=\begin{cases}1, & |u|\le g/2 \\ 0, & g/2<|u|\end{cases}$, $\chi(u)=\begin{cases}0, & |u|\le g/2 \\ \pi/2, & g/2<|u|\end{cases}$

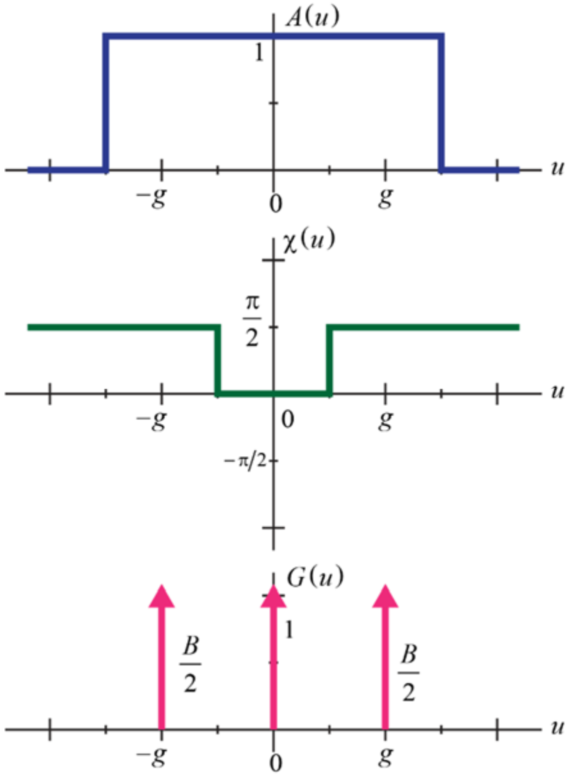
	-g	0	g
$F(u)$	$i\dfrac{B}{2}$	1	$i\dfrac{B}{2}$
$A(u)$	0	1	0
$e^{-i\chi(u)}$	$-i$	1	$-i$
$G(u)$	0	1	0



$$G(x)=0\cdot e^{-2\pi igx}+1+0\cdot e^{2\pi igx}=\underline{\underline{1}}$$

b) $A(u)=\begin{cases}1, & |u|\le 3g/2\\ 0, & 3g/2<|u|\end{cases}$, $\chi(u)=\begin{cases}0, & |u|\le g/2\\ \pi/2, & g/2<|u|\end{cases}$

	-g	0	g
$F(u)$	$i\dfrac{B}{2}$	1	$i\dfrac{B}{2}$
$A(u)$	1	1	1
$e^{-i\chi(u)}$	$-i$	1	$-i$
$G(u)$	$\dfrac{B}{2}$	1	$\dfrac{B}{2}$

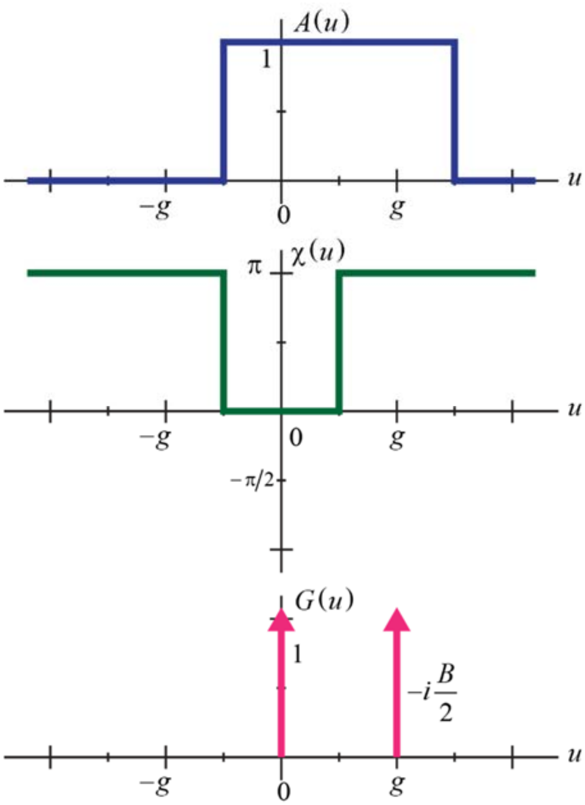


$$G(x)=\frac{B}{2}\cdot e^{-2\pi igx}+1\cdot e^{-i0}+\frac{B}{2}\cdot e^{2\pi igx}$$

$$G(x)=1+B\cdot\left(\frac{e^{2\pi igx}+e^{-2\pi igx}}{2}\right)=\underline{\underline{1+B\cdot\cos(2\pi gx)}}$$

c) $A(u)=\begin{cases}1, & |u-g/2|\le g\\ 0, & g<|u-g/2|\end{cases}$, $\chi(u)=\begin{cases}0, & |u|\le g/2\\ \pi, & g/2<|u|\end{cases}$

	-g	0	g
$F(u)$	$i\frac{B}{2}$	1	$i\frac{B}{2}$
$A(u)$	0	1	1
$e^{-i\chi(u)}$	-1	1	-1
$G(u)$	0	1	$-\frac{iB}{2}$



$$G(x)=0\cdot e^{-2\pi igx}+1\cdot e^{-i0}-\frac{iB}{2}\cdot e^{2\pi igx}$$

$$G(x)=1-\frac{iB}{2}\cdot e^{2\pi igx}$$