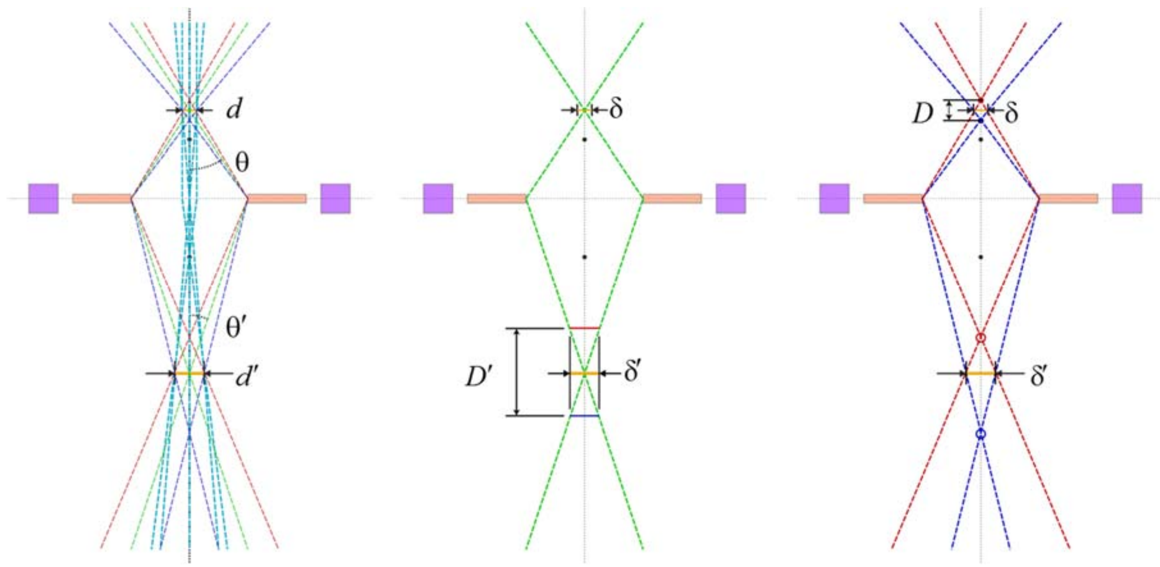
6. Electron Lenses

Depths of field & focus (I)
We found that, for an object at a distance p in front of the lens plane, an image was formed at a distance q behind the lens plane. But real objects are not perfectly flat. Some of the most interesting things that we may want to image will extend some distance along the optic axis. So, the image will also be spread along the optic axis. We need to decide how much of the object and how much of the image are in focus.

First consider an object of lateral size d, which produces an image of lateral size $d' = M \cdot d$. We can verify the size and location of the image by following at least two rays that pass through various points on the object. Rays parallel to the optic axis in front of the lens will pass through the focal point in the back of the lens. Rays passing through the center of the lens will continue undeflected in the back of the lens.



Depths of field & focus (II)
Next consider rays from a point source on the optic axis in the object plane. In particular, look at rays that just glance the aperture on either side in the lens plane, making the highest angle possible with respect to the optic axis. If this maximum angle in front of the lens is θ, the maximum angle in the back of the lens is $\theta' = \theta/M$. If we take the image in different planes within a range of height D' around the correct image plane, it will be broadened to a width less than $D' \cdot \tan\theta' \approx D' \cdot \theta'$.

Let's say the image of any point on the object is broadened to a lateral size δ', for some reason. Then the corresponding smallest resolvable feature size in the object plane is $\delta = \delta'/M$. Now, if we take the image in some other plane, the lateral broadening (due to being out of focus) is $D' \cdot \theta'$. So, we can consider the image to be in focus in these other planes if $D' \cdot \theta' < \delta'$, or $D' < M^2 \cdot \delta/\theta$. The vertical distance over which the image remains in focus is called the depth of focus

$$\text{depth of focus} = M^2 \cdot \frac{\delta}{\theta}$$

Depths of field & focus (III)
The images of points on the optic axis that are slightly above and below the object plane will be formed slightly above and below the image plane, respectively. In the correct image plane, the images of these points will be out of focus, i.e., broadened. Say the vertical separation between the object points is D.

Their lateral separation in the correct object plane is $D \cdot \theta$ and in the correct image plane it is $M \cdot D \cdot \theta$. So, if $M \cdot D \cdot \theta < \delta'$, these points are both in focus. Therefore, both points are in focus if $D < \delta/\theta$. The limiting value – the maximum vertical separation for which the object remains in focus – is

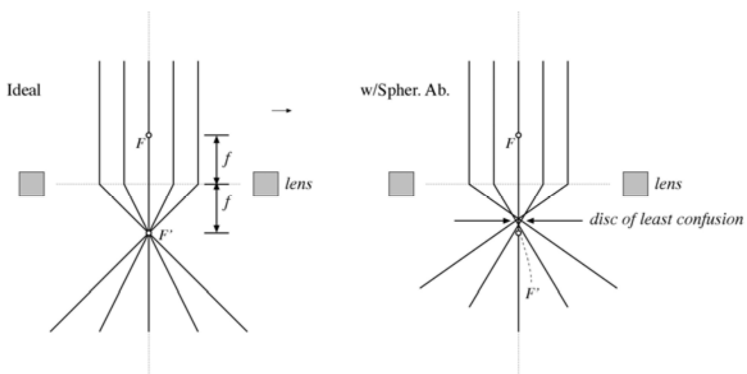
$$\text{depth of field} = \frac{\delta}{\theta}$$

Notice that

$$\text{depth of focus} = M^2 \cdot \text{depth of field}$$

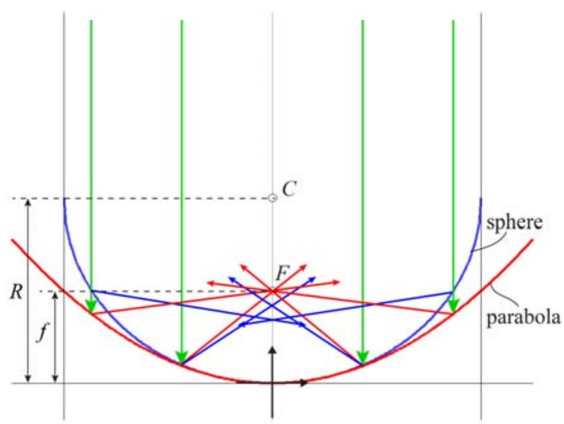
Spherical aberration

We assumed an ideal lens, which focuses all rays parallel to the optical axis in front of the lens to the focal point in the back of the lens. But many real lenses, especially electron lenses, do not perfectly focus all of these rays to the focal point. Essentially, the focal length becomes shorter for rays further from the optic axis, which smears out the focused spot along the optic axis. This is called spherical aberration, because it is a characteristic of a lens formed with a spherical curvature, rather than the parabolic curvature that an ideal lens would have. One consequence is a "disk of least confusion", which is the region where the focused spot reaches its minimum size, located somewhere between the focal point for rays very close to the optic axis and the lens plane.



Parabolic approximation of a sphere

Let's compare the focusing properties of a sphere and a parabola. It is easiest to analyze concave reflecting mirrors, which act like lenses, rather than the more common refracting, convex lenses.



The equation for a sphere of radius R, center on the y-axis, with its lowest point at $y = 0$ is:

$$y - R = \sqrt{R^2 - x^2}$$

Near the origin, an expansion of the equation to lowest-order is quadratic: $y \approx x^2/2R$.

A parabola is the locus of points equidistant from a line (the directrix) and a point (the focus). The equation of a parabola satisfying the criteria we used above (for the sphere) is:

$$y + f = \sqrt{(y - f)^2 + x^2}$$

or $y = x^2/4f$. So, if we choose the sphere to match the parabola near its lowert point, we need to pick the radius such that $R = 2f$. In other words, for rays very close to the optic axis, a spherical lens acts like a parabolic lens with focal length $f = R/2$.

Origin of spherical aberration

Say we did decide to use a concave, spherical mirror as a lens. We can consider the focal length to be the distance from the lens at which a ray parallel to the optic axis intersects the optic axis. Consider a reflected ray that makes an angle θ from the optic axis. Then, using a little trig, the focal length for a ray at this angle is:

$$f(\theta) = \frac{R}{2}\left[1 - \left(\frac{1}{\cos(\theta/2)} - 1\right)\right] = f_0 - \Delta f(\theta)$$

Here $f_0 = R/2$ is the focal length for rays very near the optic axis (small θ). If θ is small but non-zero, we can get a polynomial expression for $\Delta f(\theta)$ by expanding to lowest order.

$$\Delta f(\theta) = \frac{R}{2}\cdot\left(\frac{1}{\cos(\theta/2)} - 1\right) \approx \frac{R}{2}\cdot\left\{\left[\cancel{1} + \frac{1}{2}\left(\frac{\theta}{2}\right)^2\right] - \cancel{1}\right\} \approx \frac{R}{2}\cdot\frac{\theta^2}{8} = \frac{R}{16}\theta^2$$

We expect $\Delta f(\theta)$ to vary as an even power (e.g., 2) of θ, because the lens is symmetric about the y-axis. The coefficient is called the spherical aberration constant for the lens, and has units of length. For the sphere, we see that it has a particular value:

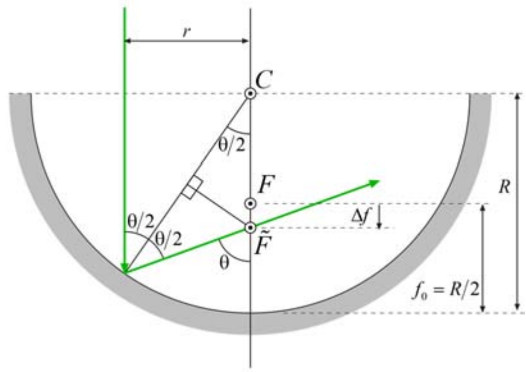
$$C_s = \frac{R}{16} = \frac{f_0}{8}$$

The spherical aberration coefficient C_s is positive, and has about the same length scale as the radius of curvature of the lens, which is comparable to the focal length. This is typically true of electron lenses. In general, we expect the focal length for rays near the optic axis (paraxial rays) to vary as:

$$f(\theta) = f_0 - C_s\theta^2$$

Alternatively, we could relate the focal length to the distance r of a ray parallel to the optic axis from the optic axis. We are dealing with small angles, so $\theta \approx r/f_0$, thus:

$$\Delta f(r) = C_s\left(\frac{r}{f_0}\right)^2$$



Effect of Δf in image plane
Say our object as at position p_0, and as $\theta \to 0$ our lens has focal length f_0 and our image is formed at q_0, so

$$\frac{1}{p_0}+\frac{1}{q_0}=\frac{1}{f_0}$$

Now say our lens is not ideal. Our object is still at p_0, but rays with higher angle will have focal length $f_0-\Delta f$, and their image will form at $q_0+\Delta q$. We then have

$$\frac{1}{p_0}+\frac{1}{q_0+\Delta q}=\frac{1}{f_0-\Delta f}$$

Now let's assume the correction is small: $|\Delta f| \ll f_0$. Some terms cancel, leaving:

$$-\frac{\Delta q}{q_0{}^2}\approx\frac{\Delta f}{f_0{}^2}$$

and finally

$$\Delta q=-\left(\frac{q_0}{f_0}\right)^2\cdot\Delta f$$

The lens is a little stronger for $\Delta f>0$, so the image will form a little closer to the lens plane $\Delta q<0$.

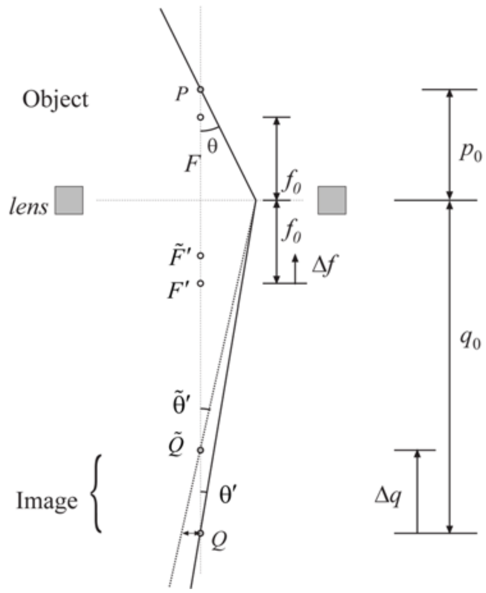
Effect of spherical aberration on resolution
Spherical aberration will cause a point on the object to be imaged as a larger, blurred feature. In TEM, we are mostly interested in the main imaging lens, called the objective lens, because it requires a large collection angle and must have a small C_s. It also contributes a large portion of the magnification used. We saw that $M=q/p$. From this we can show that $M=(q/f)-1$, so the limit $M\to\infty$ leads to $M\approx q/f$. From the preceding section, we then have

$$\Delta q\approx -M^2\cdot\Delta f=-M^2C_s\theta^2$$

A ray passing at an angle θ through a point-like object on the optic axis will cross the axis in back of the lens at $q_0+\Delta q$. When it reaches the usual, so-called "Gaussian" image plane at q_0, it will be a distance

$$\delta' \approx -\theta' \cdot \Delta q = MC_s\theta^3$$

from the axis, so this gives an estimate of the blurring of the image due to spherical aberration.



We usually are more concerned with how C_s limits the resolution of the microscope. The apparent size of our object point is found by referring the amount of spreading back to the object size:

$$\delta = \frac{\delta'}{M} = C_s\theta^3$$

Optimal β

We had earlier discussed the Rayleigh criterion as an estimate of resolution. There, we found that diffraction limited the smallest resolvable feature size, giving $\delta_d = (0.61)\lambda/\beta$, where β is the semi-angle of collection of the lens. This implies that we can always improve resolution (decrease δ) by increasing β. That would be true if our lens were ideal. But we have now found that rays at collected at high angles blur the image, due to spherical aberration. If we take the highest angle rays included in the image (angle β), the resolution limit is $\delta_s = C_s \cdot \beta^3$. This implies that the image can become worse if we increase β. So we must balance the two effects.

Let's define some reference quantities:

$$\delta_0 = \left(C_s\lambda^3\right)^{1/4}, \text{ and } \beta_0 \equiv (\lambda/C_s)^{1/4}$$

We can rewrite our resolution expressions as:

$$\delta_d = (0.61)\times\delta_0\times\left(\frac{\beta_0}{\beta}\right), \text{ and } \delta_s = \delta_0\times\left(\frac{\beta}{\beta_0}\right)^3$$

It is not clear exactly how to combine these two terms, but if we assume they are small, and the contributions to each are independent, a good guess is:

$$\delta_{net} \approx \sqrt{\delta_d^{\,2} + \delta_s^{\,2}}$$

Now we can minimize δ_{net} with respect to β:

$$\left.\frac{d\delta_{\text{net}}}{d\beta}\right|_{\beta=\beta_{\text{opt}}} = 0$$

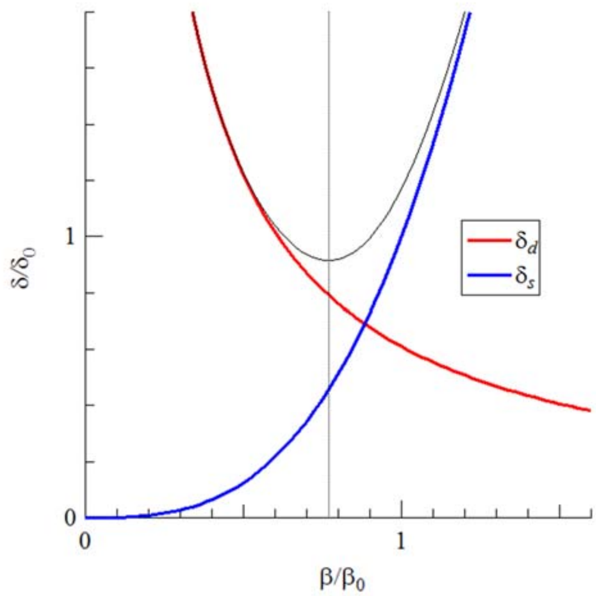
We first find the optimal value of β:

$$\beta_{\text{opt}} = \frac{(0.61)^{1/4}}{3^{1/8}} \cdot \beta_0 = (0.77) \cdot \beta_0$$

Plugging that back in, we get a new estimate for the resolution of this lens:

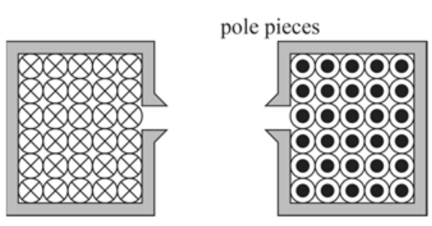
$$\delta_{\min} = (0.61)^{3/4} \cdot \sqrt{3^{1/4} + 3^{-3/4}} \cdot \delta_0 = (0.91) \cdot \delta_0$$

This is called the "practical" resolution of the lens.

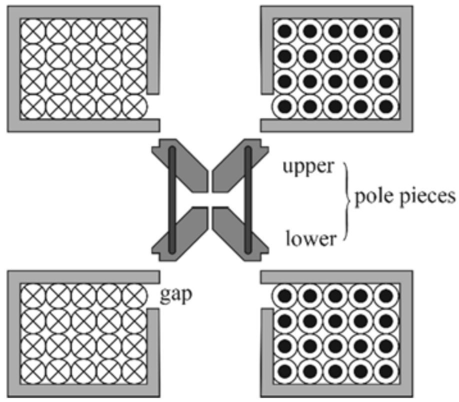


Electromagnetic lenses

Most electron lenses are current-carrying coils wound in loops around an axis. Each lens is usually encased in a ferrous (magnetizable) shell, called the pole-piece, with a hole in the middle, called the bore. There are gaps in the pole piece along the bore, which allows the magnet field lines to concentrate into the region where the electron beam is traveling. for a large lens, the current produces heat, so the lenses usually have water cooling built in.

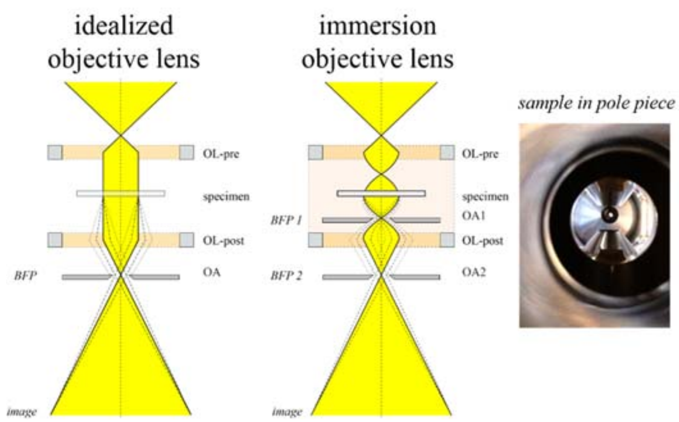


A common configuration is the twin lens, which is split into two separated halves. A removable pole-piece insert, possibly with both upper and lower segments, is set in the gap between the halves. This pole piece is tailored to meet certain requirements of the instrument, and maybe be exchangeable for different applications.



Immersion lens
The twin lens we described is almost certainly an immersion-type lens, which may be a familiar concept from optical microscopy. This means that the sample is actually immersed in the lens medium, which is usually an oil for an optical microscope. For an electron lens, the specimen is immersed in the lens's magnetic field.

For TEM imaging, we are often most interested in the part of the lens below the sample - the post-specimen lens. For that reason, we often idealize the lens by ignoring the immersion aspect and assuming that the specimen is in a field-free region, so that there is no deflection of rays inside the lens. The pre-specimen lens may play a secondary role in forming the probe that illuminates a specimen.

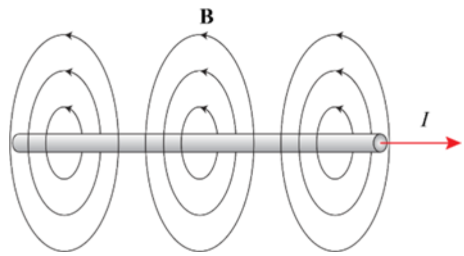


In an immersion lens, rays are continually being focused through the lens volume, so we may actually have beam crossover inside the lens. This allows the use of in-lens apertures to affect image contrast. On the down side, a magnetic sample may be hard to image with this kind of lens, and may even become attached to the pole piece if it is not tightly secured in the specimen holder.

Magnetic fields and forces
Magnetic fields are produced by electrical currents, which consist of moving electrical charges, at least in the macroscopic work of electrical circuits. Ampere's law can be used in some cases to find the magnetic field around a current carrying wire.

$$\oint \mathbf{B} \cdot ds = \mu_0 \cdot I$$

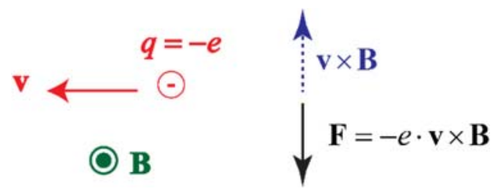
For a straight wire, it tells us that the field should form closed loops around the wire; it doesn't point toward or away from the wire, or even parallel to the wire.



Magnetic fields create forces on moving electric charges. The force on a point charge is:

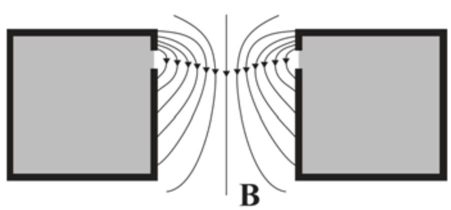
$$\mathbf{F} = q \cdot \mathbf{v} \times \mathbf{B}$$

where $\mathbf{v}$ is the velocity. The cross-product always points perpendicular to both factors. That is, $\mathbf{F}$ is always perpendicular to $\mathbf{v}$ and $\mathbf{B}$ (unless $\mathbf{v}$ and $\mathbf{B}$ are parallel, in which case $\mathbf{F}$ vanishes.) This is called the Lorentz force law.

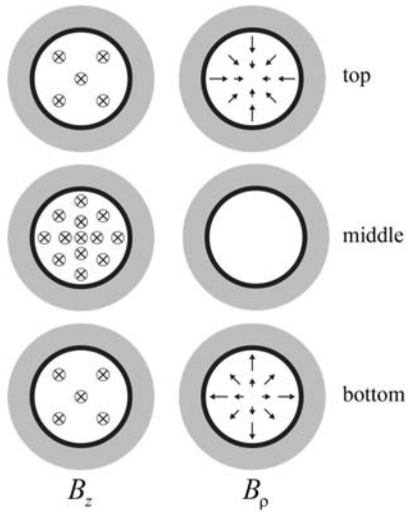


Lens field

What type of magnetic field can we expect in one of these lenses? If the lens is symmetric about its axis (axial symmetry), we expect only axial and radial components. (More on this later.) For a single lens with a single gap in the pole piece, we expect a strong axial component near the gap. Magnetic field lines form closed loops, so we can draw lines curving towards the axis at the top of the lens, along the axis near the gap, and away from the axis at the bottom of the lens. (The directions are reversed if the lens current is reversed.) This is enough to draw a simple sketch of the field lines through various slices normal to the lens axis.



We have used B for the magnetic field, and cylindrical coordinates with components ρ, ϕ, and z.



Conditions on magnetic field

At some point, we actually need a theoretically description of the magnetic field inside an electron lens. No one has ever detected a magnetic monopole, so one of Maxwell's equations tells us $\vec{\nabla}\cdot\mathbf{B}=0$, meaning that the field has no divergence. The divergence in cylindrical coordinates is:

$$\vec{\nabla}\cdot\mathbf{B}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho)+\frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi}+\frac{\partial B_z}{\partial z}$$

From the axial symmetry of the lens, we know that $B_\phi=0$, so this leaves us with

$$\frac{\partial}{\partial\rho}(\rho B_\rho)=-\rho\frac{\partial B_z}{\partial z}$$

We can write the integral over ρ of this equation:

$$\int_\rho\frac{\partial}{\partial\rho}(\rho B_\rho)\cdot d\rho=-\int_\rho\rho\cdot\left(\frac{\partial B_z}{\partial z}\right)\cdot d\rho$$

Our lives get easier if we assume that the derivative of B_z only depends on z:

$$\frac{\partial B_z}{\partial z}(\rho,z)=\frac{\partial B_z}{\partial z}(z)$$

Now we can do the integral

$$\rho B_\rho=-\frac{\rho^2}{2}\frac{\partial B_z}{\partial z}$$

So this all reduces to a simple relation between B_z and B_ρ:

$$B_\rho=-\frac{\rho}{2}\frac{\partial B_z}{\partial z}$$

Model: Bell-shaped field ("Glockenfeld")

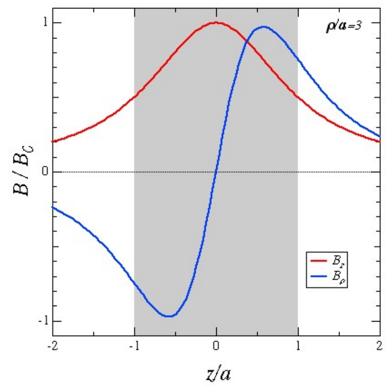
People have modeled the magnetic fields in electron lenses for many decades. The problem is not hard if we assume we know B_z. One favored model is a bell-shaped field, where B_z follows the equation:

$$B_z(z) = \frac{B_0}{1 + \left(\frac{z}{a}\right)^2}$$

We are assuming the middle of the lens is at $z = 0$. From this, we can quickly find B_ρ.

$$B_\rho(\rho, z) = \frac{\rho z}{a^2} \cdot \frac{B_z(z)}{1 + \left(\frac{z}{a}\right)^2}$$

Clearly B_ρ switches sign between $z < 0$ and $z > 0$, which is what we had assumed earlier.



Model: Uniform B_z in lens

Here is another simple model: Assume the field inside the lens points uniformly up or down along z, but vanishes everywhere else. If the lens spans the range $-a < z < a$, we can write:

$$B_z = B_0 \cdot [u(z+a) - u(z-a)]$$

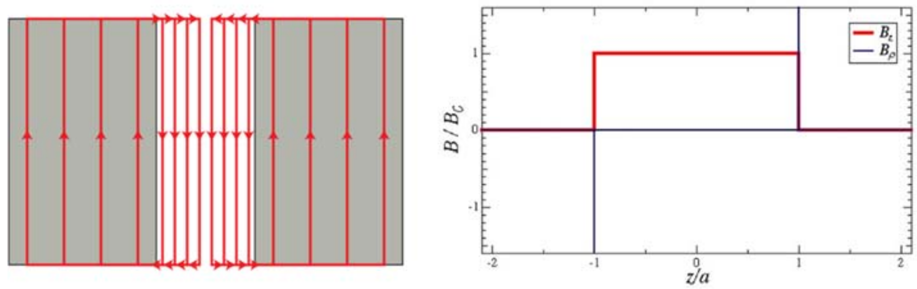
where $u(z)$ is the step function

$$u(z) = \begin{cases} 1, & z < 0 \\ 0, & z > 0 \end{cases}$$

We can find the B_ρ using our previous expression. The derivative of a step function is a delta (δ) function, so:

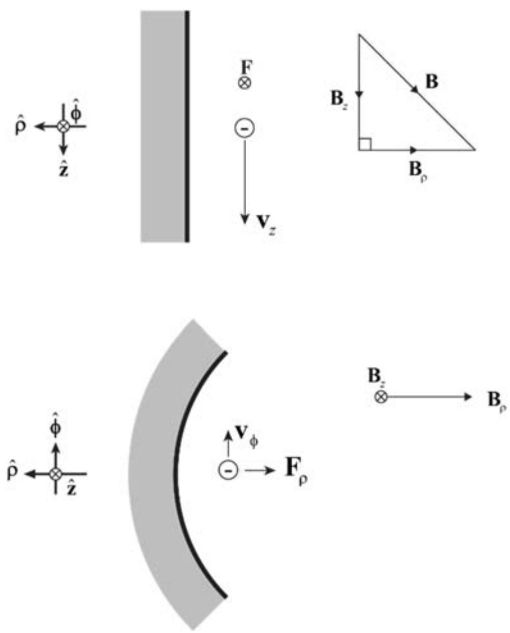
$$B_\rho = -\frac{\rho B_0}{2} \cdot \delta(z+a) + \frac{\rho B_0}{2} \delta(z-a)$$

The radial component has a spike in one direction at $z = -a$ and another spike in the opposite direction at $z = a$. Sketching this out, we see that the field lines do form loops, as expected. That does not mean it will be very easy to make a lens with this type of field, only that it is theoretically possible.



Focusing action

It is not yet clear that these electromagnetic lenses focus at all. We know that an electric charge in an electric field experiences a force, and a *moving* electric charge in a magnetic field can also experience a force. So let's look at an electron moving vertically downward, parallel to the axis of our lens. In a side view, we see that the z component has no effect on the electron, since $\mathbf{v}$ is parallel to $\hat{\mathbf{z}}$. But the strong radial component of $\mathbf{B}$ near the entrance of the lens will accelerate the electron in the ϕ direction. So very soon after entering the lens, the electron will have some ϕ component of velocity.



Now, the z field (which is the strongest component) will then cause the electron to accelerate in the radial direction, towards the lens axis. You may recognize this as a a type of focusing. While proceeding down the lens, the electron will spiral in a helical trajectory about some line some line parallel to the lens axis. This is referred to as cyclotron motion. It turns out, a set of rays running parallel to the optic axis will intersect (be focused, if you will) at some point on the axis - in effect, a focal point.

Force on moving electron

We can be a bit more exact about the motion of an electron in the lens field. In cylindrical coordinates, the position of the electron is written:

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

We assumed our magnetic field had no ϕ component: $\mathbf{B} = B_\rho\hat{\boldsymbol{\rho}} + B_z\hat{\mathbf{z}}$. The force cross product can be treated like a matrix determinant:

$$\mathbf{F} = q\begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ v_\rho & v_\phi & v_z \\ B_\rho & B_\phi & B_z \end{vmatrix} = -e\begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \dot{\rho} & \rho\dot{\phi} & \dot{z} \\ B_\rho & 0 & B_z \end{vmatrix} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\varphi}} + F_z\hat{\mathbf{z}}$$

So we can now sort out all of our various force components:

$$F_\rho = -eB_z\rho\dot{\phi}$$
$$F_\phi = -eB_\rho\dot{z} + eB_z\dot{\rho}$$
$$F_z = eB_\rho\rho\dot{\phi}$$

Equations of motion

Now we apply Newtons Law: $\mathbf{F} = m\mathbf{a}$, i.e.,

$$\mathbf{F} = m\ddot{\mathbf{r}} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\varphi}} + F_z\hat{\mathbf{z}}$$

Some of this is easier in cartesian (rectangular) coordinates (x , y , z). The transformation is:

$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

So our first derivatives are:

$$\dot{\hat{\boldsymbol{\rho}}} = \dot{\phi}\hat{\boldsymbol{\phi}},\ \dot{\hat{\boldsymbol{\phi}}} = -\dot{\phi}\hat{\boldsymbol{\rho}} \text{, and } \dot{\hat{\mathbf{z}}} = \dot{\hat{\mathbf{z}}}$$

The second derivatives are:

$$\ddot{\hat{\boldsymbol{\rho}}} = -\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{\phi}\hat{\boldsymbol{\phi}}\ ,\ \dot{\hat{\boldsymbol{\phi}}} = -\ddot{\phi}\hat{\boldsymbol{\rho}} - \dot{\phi}^2\hat{\boldsymbol{\phi}} \text{, and } \ddot{\hat{\mathbf{z}}} = \ddot{\hat{\mathbf{z}}}$$

Now we can find the velocity and acceleration in cylindrical coordinates:

$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$
$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}\right)\hat{\boldsymbol{\varphi}} + \ddot{z}\hat{\mathbf{z}}$$

There is another way to write the acceleration

$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\dot{\phi}\right)\hat{\boldsymbol{\varphi}} + \ddot{z}\hat{\mathbf{z}}$$

After all that, we can multiply by mass to get each component of force:

$$m\ddot{\rho} - m\rho\dot{\phi}^2 = F_\rho = -eB_z\rho\dot{\phi}$$
$$m\cdot\frac{d}{dt}\left(\rho^2\dot{\phi}\right) = \rho F_\phi = -eB_\rho\rho\dot{z} + eB_z\rho\dot{\rho}$$
$$m\ddot{z} = F_z = eB_\rho\rho\dot{\phi}$$

Next, we will analyze each component separately, using our uniform field model for the lens.

Solve (I): ϕ component

Let's start with ϕ. We had expressions for B_ρ and B_z , so

$$m\cdot\frac{d}{dt}\left(\rho^2\dot{\phi}\right) = \frac{e\rho^2B_0\dot{z}}{2}\cdot[\delta(z+a) - \delta(z-a)] + e\rho\dot{\rho}B_0\cdot[u(z+a) - u(z-a)]$$

where we have used the fact that the derivative of $u(z)$ is $\delta(z)$. Now this can be written as

$$\frac{d}{dt}\left(\rho^2\dot{\phi}\right)=\omega_L\cdot\frac{d}{dt}\{\rho^2\cdot[u(z+a)-u(z-a)]\}$$

where the constants have been combined into a angular frequency, called the Larmor frequency

$$\omega_L=\frac{eB_0}{2m}$$

It is easiest to analyze rays with no initial rotational component, i.e., $\dot{\phi}=0$ for $z<-a$. Then

$$\dot{\phi}=\omega_L\cdot[u(z+a)-u(z-a)]=\begin{cases}0, & z<-a\\ \omega_L, & -a\le z<a\\ 0, & a\le z\end{cases}$$

This describes rotational motion inside the lens, as we expected.

Solve (II): z component

We need to look at the z component, too. Ideally, there would be no change in the velocity component along $\hat{\mathbf{z}}$ as the electron passes through the lens. Unfortunately, that is not the case. The z acceleration is:

$$\ddot{z}=-\omega_L\rho^2\dot{\phi}\cdot[\delta(z+a)-\delta(z-a)]$$

We can use a trick here. Write:

$$\ddot{z}=-(\omega_L\rho)^2\cdot[u(z+a)-u(z-a)]\cdot[\delta(z+a)-\delta(z-a)]$$

Then multiply by $\dot{z}\cdot dt$ and integrate. At the entrance of the lens ($z=-a$)

$$\int_{t=0^-}^{0^+}\ddot{z}\cdot\dot{z}\cdot dt=-\omega_L{}^2\cdot\int_{t=0^-}^{0^+}\rho^2[u(z+a)-u(z-a)]\cdot[\delta(z+a)-\delta(z-a)]\cdot\dot{z}\cdot dt$$

The electron trajectory is continuous, so $\rho|_{0^-}=\rho|_{0^+}=\rho_0$. We might notice that:

$$\frac{d}{dt}[u^2(z)]=2u(z)\cdot\delta(z)\cdot\dot{z}$$

Since $u(z)$ is just a step function, we can write $u^2(z)=u(z)$. So the integral gives

$$v_z{}^2-v_{z0}{}^2=-\omega_L{}^2\cdot\rho_0{}^2$$

where v_{z0} and v_z the electron velocity in the z-direction outside and inside of the lens, respectively. So we now know that the z-velocity inside the lens will be slightly smaller as the entrance radius gets bigger:

$$v_z=\sqrt{v_{z0}{}^2-(\rho_0\omega_L)^2}$$

Solve (III): rho component

The ρ component is the most important for focusing. Combining what we have so far:

$$\ddot{\rho}=-\omega_L{}^2\cdot[u(z+a)-u(z-a)]\cdot\rho$$

This is not hard to solve. Inside the lens, the radial motion is harmonic. We can write the solution as:

$$\rho(t)=\rho_0\cdot\cos(\omega_L\cdot t+\theta)$$

We also know the z motion is uniform inside the lens, $z=v_z'\cdot t-a$. If the electron enters the lens at radius ρ_i, and an angle θ_i with respect to the optic axis

$$\rho_0 = \sqrt{\rho_i^{\,2} + \frac{\tan^2\theta_i}{k'^2}} \text{ and } \theta = -\tan^{-1}\left(\frac{\tan\theta_i}{k'\cdot\rho_i}\right)$$

In terms of z:

$$\rho(z) = \rho_0 \cdot \cos[k'\cdot(z+a)+\theta]$$

The radius oscillates along *z* with wave number k', where

$$k' = \frac{\omega_L}{v_z'} = \frac{\omega_L}{\sqrt{v_z^{\,2} - (\rho_0\omega_L)^2}}$$

The downside here is that the wavenumber depends on the maximum radius ρ_0, so not all rays will oscillate with the same wavelength. Therefore, our lens is not ideal. We could assume ρ_0 is small and expand

$$k' \approx \frac{\omega_L}{v_z} + \frac{1}{2}\rho_0^{\,2}k^3 = k + \Delta k$$

On-axis rays will have a wavenumber k, but the wavenumber increases (focusing strength increases) quadratically as ρ_m increases. This sounds a bit like spherical aberration.

Find paraxial ray

We can draw a ray diagram for this lens. Also, assume $k' = k$ for now. If the incident ray makes an angle θ_i w.r.t. the optic axis:

$$\rho(z) = \rho_i \cdot \cos[k\cdot(z+a)] + (\tan\theta_i/k)\cdot\sin[k\cdot(z+a)]$$

The first derivative is

$$\frac{d\rho}{dz} = -k\rho_i \cdot \sin[k(z+a)] + \tan\theta_i \cdot \cos[k(z+a)]$$

Let's first consider a ray moving parallel to the optic axis ($d\rho/dz = 0$) with $\rho = \rho_i$ as it enters the lens at $z = -a$. So $\theta_i = 0$ and

$$\rho(z) = \rho_i \cdot \cos[k\cdot(z+a)]$$

At the lens exit ($z = a$), the ray has

$$\rho(z)\big|_{z=a} = \rho_i \cdot \cos(2ka) \text{ and } \left.\frac{d\rho}{dz}\right|_{z=a} = -\rho_i \cdot \sin(2ka)$$

After that ($z > a$), it moves in a straight line with the same slope, described by:

$$\rho_+(z) = \rho_i \cdot [-k\cdot\sin(2ka)\cdot(z-a) + \cos(2ka)]$$

Find focal length

The back focal point is at the location $z = z_f$ where the paraxial ray crosses the optic axis. Setting

$$\rho_+(z)\big|_{z=z_f} = 0$$

This gives the relation (keeping distances as ratios w.r.t. a):

$$\frac{z_f}{a} = 1 + \frac{1}{ka\cdot\tan(2ka)}$$

We want to define the focal length in a way that allows us to use the ideal lens equation. We can extend the paraxial ray back to where it intersects the incident ray with radius:

$$\rho_+(z)\big|_{z=z_0} = \rho_i$$

This gives the location z_0 of the "principal" plane for the back of the lens:

$$\frac{z_0}{a} = 1 - \frac{\tan(ka)}{ka}$$

The focal length is the distance from the focal point to the principal plane $f = z_f - z_0$, leading to:

$$\frac{f}{a} = \frac{z_f}{a} - \frac{z_0}{a} = \frac{\tan(ka)}{ka} + \frac{1}{ka \cdot \tan(2ka)} = \frac{1}{ka \cdot \sin(2ka)}$$

Find ray through lens center

Another special case is a ray passing through the center of the lens ($z = 0$).

$$\rho(z)\big|_{z=0} = 0$$

We find that

$$\rho(z) = -\rho_0 \cdot \sin(kz)$$

At the exit of the lens

$$\left.\frac{d\rho}{dz}\right|_{z=a} = -k\rho_0 \cos(ka)$$

Extending this ray as a straight line below the lens ($z > a$), the equation is

$$\rho_+(z) = -\rho_0 \cdot [\sin(ka) + k\cos(ka) \cdot (z-a)]$$

We can trace this line back to the optic axis using

$$\rho_+(z)\big|_{z=z_n} = 0$$

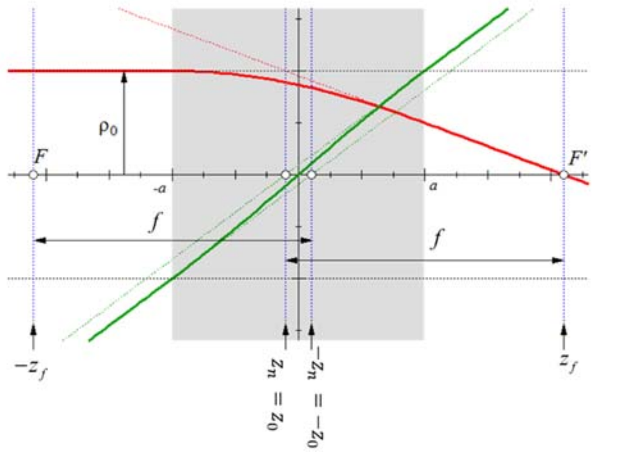
the intersection gives the location z_n of the "nodal" plane for the back of the lens

$$\frac{z_n}{a} = 1 - \frac{\tan(ka)}{ka}$$

Interestingly, the nodal and principal planes coincide ($z_p = z_n$), which is not true for all lenses.

Ray diagram: Uniform B inside lens

A graphical illustration of what we have just derived is shown below.



Electron trajectory: Uniform B lens

The actual motion of an electron passing through the lens may be different than expected. We have some equations of motion

$$\frac{d^2\rho}{dz^2} = -k^2\rho \text{, and } \frac{d\phi}{dz} = k$$

For a paraxial ray

$$\rho(z) = \rho_0 \cdot \cos[\phi(z)] \text{ and } \phi(z) = k \cdot (z+a)$$

Let's change to Cartesian coordinates:

$$x(z) = \rho(z) \cdot \cos[\phi(z)] = \frac{1}{2}\rho_0 \cdot \{1 + \cos[2\phi(z)]\}$$

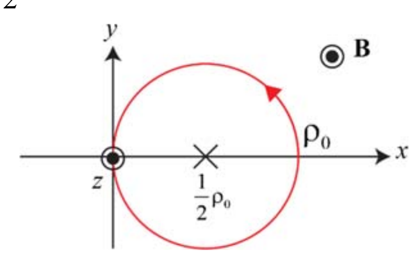
$$y(z) = \rho(z) \cdot \sin[\phi(z)] = \frac{1}{2}\rho_0 \cdot \sin[2\phi(z)]$$

This describes an orbit centered at $x = \rho_0/2$. The angular motion (the rotation of the image) actually changes as $2\phi(z)$, twice as fast we might have thought. The wavelength is

$$2\phi(z)\big|_{z=-a+\lambda} = 2\pi \to \phi(z)\big|_{z=-a+\lambda} = \pi \to k\lambda = \pi \to \lambda = \frac{\pi}{k} = \frac{v_z \cdot \pi}{\omega_L}$$

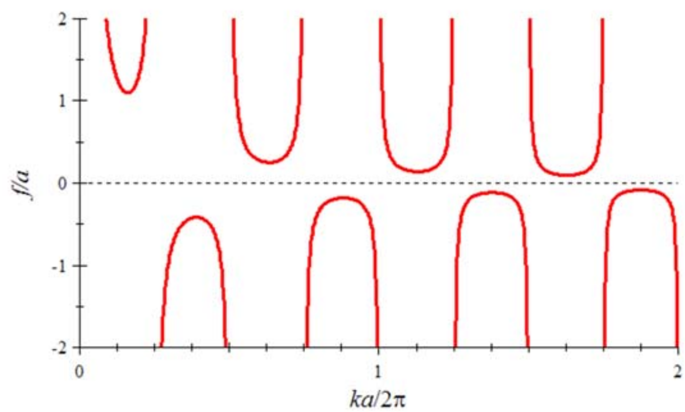
(This is the wavelength for the orbital motion, not the electron's wavelength.) The period for one orbit can be found from the frequency. If T_L is the period of oscillation at the Larmor frequency, the period for this motion is:

$$f = \frac{v_z}{\lambda} = \frac{\omega_L}{\pi} = \frac{\omega}{2\pi} = \frac{1}{T} \to T = \frac{T_L}{2}$$



The electron is undergoing cyclotron resonance in this case, and the angular frequency $\omega = eB_0/m$ is called the cyclotron frequency. That is, for an electron with a transverse velocity v, the Lorentz force is $ma = evB_0$. This is a centripetal force, so $v = \omega r$ and $a = \omega^2 r$, giving $\omega = eB_0/m$.

Focal length
We derived an equation for the focal length of this electromagnetic lens. Note that k increases as the magnetic field increases, so the oscillation frequency gets higher. We may expect a stronger field to produce a stronger lens, with a shorter focal length: That is not always true. Surprisingly, f decreases and increases cyclically as k is increased, though the minimum get smaller with every cycle. This type of oscillation occurs because parallel rays may go through multiple crossovers within the lens, so the focal point moves closer and then farther from the exit plane of the lens.



Notice that the focal length is negative when $\sin(2ka)$, in which case the image of an object in front of the lens ($p > 0$) will be virtual ($q < 0$), in this case.

Estimate spherical aberration coefficient
Let us see if we can find a C_s for this lens. An expansion in terms of k gives:

$$f \approx f|_{k'=k} + \left.\frac{\partial f}{\partial k}\right|_{k'=k} \cdot \Delta k = f_0 - \Delta f$$

The two expansion coefficients are

$$f|_{k'=k} = \frac{1}{k \cdot \sin(2ka)} \text{ and } \left.\frac{\partial f}{\partial k}\right|_{k'=k} = -f_0 \cdot \left[\frac{1}{k} + \frac{2a}{\tan(2ka)}\right]$$

It will be easiest if we just look at a case similar to the conditions where we might actually be using the lens. Most likely (at least for the objective lens) we want strong excitation, with a short focal length, so let's assume $\sin(2ka) = 1$. Then

$$f|_{k'=k} \approx \frac{1}{k} \text{ and } \left.\frac{\partial f}{\partial k}\right|_{k'=k} \approx -\frac{f_0}{k}$$

Now we can say

$$\Delta f = \frac{f_0}{k} \cdot \Delta k = \frac{\rho_0^{\,2}}{2 f_0}$$

To find C_s , we need to relate this to angle. At high magnification, $\theta \approx \rho_0 / f_0$, so we can write

$$\Delta f \approx \frac{1}{2} \cdot f_0 \cdot \theta^2 = C_s \cdot \theta^2$$

Now we can write an expression for C_s

$$C_s \approx \frac{f_0}{2} = \frac{v_z}{2\omega_L} = \frac{mv_z}{eB_0}$$

This is even worse (bigger) than for a spherical lens, which we previously showed had $C_s = f_0/8$!