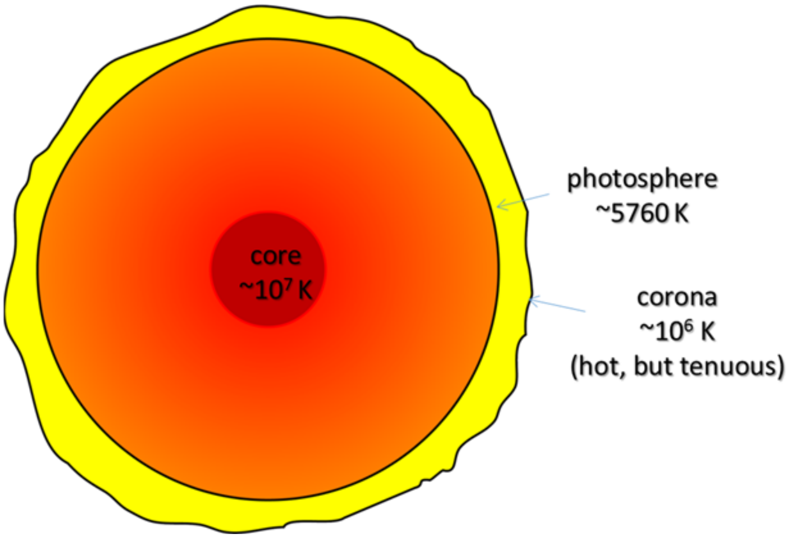
2. <u>Solar Spectrum</u>

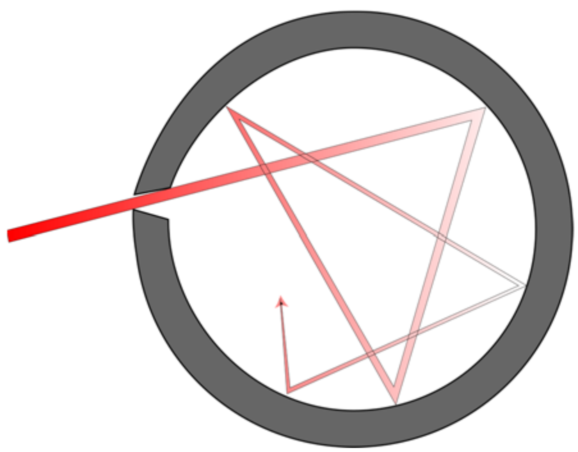
The sun as a blackbody

The photosphere (where most of the light originates) has a temperature of approximately 6000 K. Understanding the spectra of hot, radiant objects was critical to modern physics. Perfect emitters/absorbers are called "blackbodies"



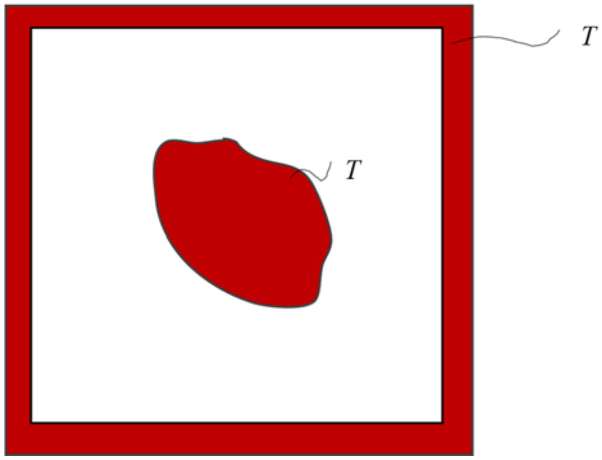
Blackbody

A blackbody could consist of a cavity with tiny hole. Radiation entering a blackbody has low probability of escaping. In steady state (constant T), the rate of energy entering must equal the rate of energy escaping,



Blackbody in a cavity

Allow the blackbody to reach equilibrium with the cavity. The BB must be radiating at the same rate it is absorbing. Emitted spectrum must be independent of orientation of BB. Spectrum of a BB must depend on T only!



Photons

Based on Planck's formulation, Einstein proposed that photons have energy:

$$E = hf$$

For any wave:

$$v = \lambda f$$

For light in vacuum:

$$c = \lambda f$$

So

$$E = \frac{hc}{\lambda}$$

Recall $hc = 1240 \text{ eV} \cdot \text{nm}$ (e.g., $\lambda = 620 \text{ nm}$, $E = 2.0 \text{ eV}$). A harmonic, traveling wave is described by its wave function, either a sinusoid

$$\psi(x,t) = A \cdot \cos\left[2\pi\left(\frac{x}{\lambda} - ft\right) + \phi\right]$$

or a complex exponential

$$\psi(x,t) = A \cdot \exp\left[2\pi i\left(\frac{x}{\lambda} - ft\right)\right]$$

We can use the wave number:

$$k = \frac{1}{\lambda}$$

sometimes defined as

$$k = \frac{2\pi}{\lambda}$$

De Broglie's hypothesis:

$$p = \frac{h}{\lambda}$$

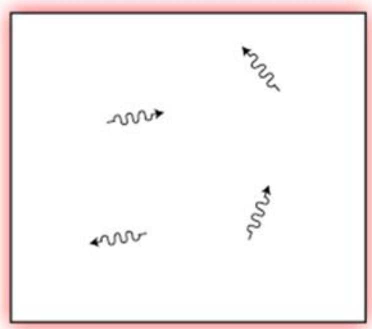
So, for photons:

$$E = pc$$

Blackbody radiation: approach

Assume a cavity and radiation within it have temperature T . To find the # of photons per unit photon energy, per unit volume we need to know: i) density of states (# of states/unit volume/unit energy) and ii) distribution function (average # of photons in each state).

$$n = \frac{N}{V}$$



$$\frac{dN}{dE} = D(E) \cdot f(E)$$

$$\frac{dn}{dE} = D(E) \cdot f(E)/V$$

Blackbody radiation: DOS(I)

First we find $D(E)$. The waves are harmonic, i.e.,

$$\psi(x,t) = \psi(x) \cdot e^{-2\pi ift}$$

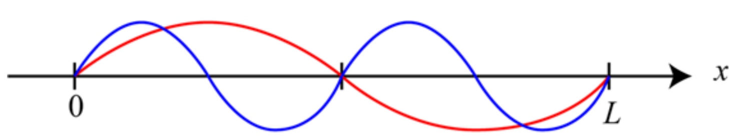
so we only need to consider the spatial part. In 1-D:

The trick is to assume periodic boundary conditions:

$$\psi(x+L) = \psi(x) \cdot e^{2\pi ik/L}$$

Then the allowed wave numbers are $k_n = n/L$, so

$$\lambda_n = \frac{L}{n}$$

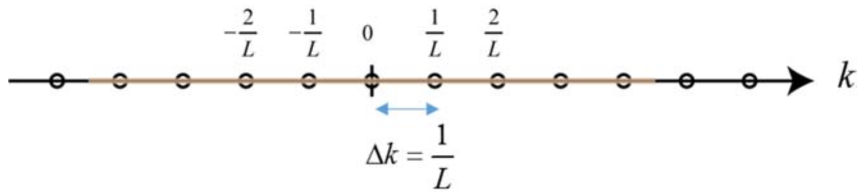


Blackbody radiation: DOS (II)

We can enumerate the allowed wavenumbers

$$k = 0, \pm\frac{1}{L}, \pm\frac{2}{L}, \ldots$$

These are equally spaced along the number line.

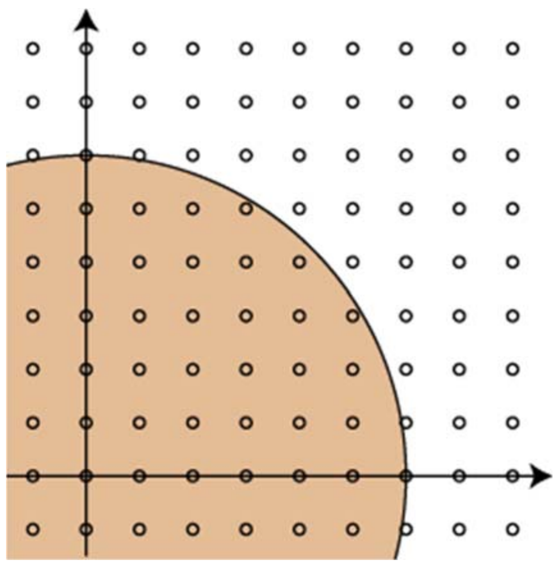


There are g states per $1/L$ in wave number. The # of states with $|k_n| < k$ is

$$N(k) = \frac{g \cdot (2k)}{\left(\frac{1}{L}\right)} = 2gkL$$

In 2-D, we would have

$$N(k) = \frac{g \cdot (\pi k^2)}{\left(\frac{1}{L^2}\right)} = \pi g k^2 L^2$$



In 3-D, we end up with

$$N(k) = \frac{g \cdot \left(\frac{4}{3}\pi k^3\right)}{\left(\frac{1}{L^3}\right)} = \frac{4}{3}\pi g k^3 L^3$$

Blackbody radiation: DOS (III)

We need to change to an energy basis, using $E = hck$

Now we find the DOS

$$D(E) = \frac{dN}{dE}$$

In 1-D, this gives a constant D(E)

$$N(E) = \frac{2gL}{hc} \cdot E \ , \ D(E) = \frac{2gL}{hc}$$

In 2-D, D(E) is linear

$$N(E) = \frac{\pi g L^2}{h^2 c^2} \cdot E^2 , \; D(E) = \frac{2\pi g L^2}{h^2 c^2} \cdot E$$

In 3-D, D(E) is quadratic

$$N(E) = \frac{4\pi g L^3}{3h^3 c^3} \cdot E^3 , \; D(E) = \frac{4\pi g L^3}{h^3 c^3} \cdot E^2$$

Blackbody spectrum (I)

Electron spin gives $g = 2$. We can specify $V = L^3$. The # of photons/photon energy/unit volume is

$$\frac{dn}{dE} = \frac{D(E) \cdot f(E)}{V} = \frac{8\pi E^2}{h^3 c^3 (e^{E/kT} - 1)}$$

This is Planck's radiation law. The energy/photon energy/unit volume is

$$\frac{du}{dE} = E \cdot \frac{dn}{dE} = \frac{8\pi E^3}{h^3 c^3 (e^{E/kT} - 1)}$$

This expression describes the blackbody spectrum.

Blackbody spectrum (II)

Now let's find total toal number of photons per unit volume:

$$n = \int_{E=0}^{\infty} dE \cdot \frac{dn}{dE} = \int_{E=0}^{\infty} dE \cdot \left[\frac{8\pi E^2}{h^3 c^3 (e^{E/kT} - 1)} \right] = \frac{8\pi k^3 T^3}{h^3 c^3} \cdot I_2$$

where

$$I_k = \int_{x=0}^{\infty} \frac{x^k}{e^x - 1} \cdot dx = \Gamma(k+1) \cdot \zeta(k+1)$$

Some known constants are

$$\Gamma(k+1) = k! \text{ and } \zeta(k+1) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^{k+1}$$

So $I_2 = \Gamma(3) \cdot \zeta(3) \approx (2!) \cdot (1.202) = 2.4$. Now we have

$$n \approx \frac{(60.4) k^3 T^3}{h^3 c^3}$$

Now let's fine the total energy/volume:

$$u = \int_{E=0}^{\infty} dE \cdot \frac{du}{dE} = \int_{E=0}^{\infty} dE \cdot \left[\frac{8\pi E^3}{h^3 c^3 (e^{E/kT} - 1)} \right] = \frac{8\pi k^4 T^4}{h^3 c^3} \cdot I_3$$

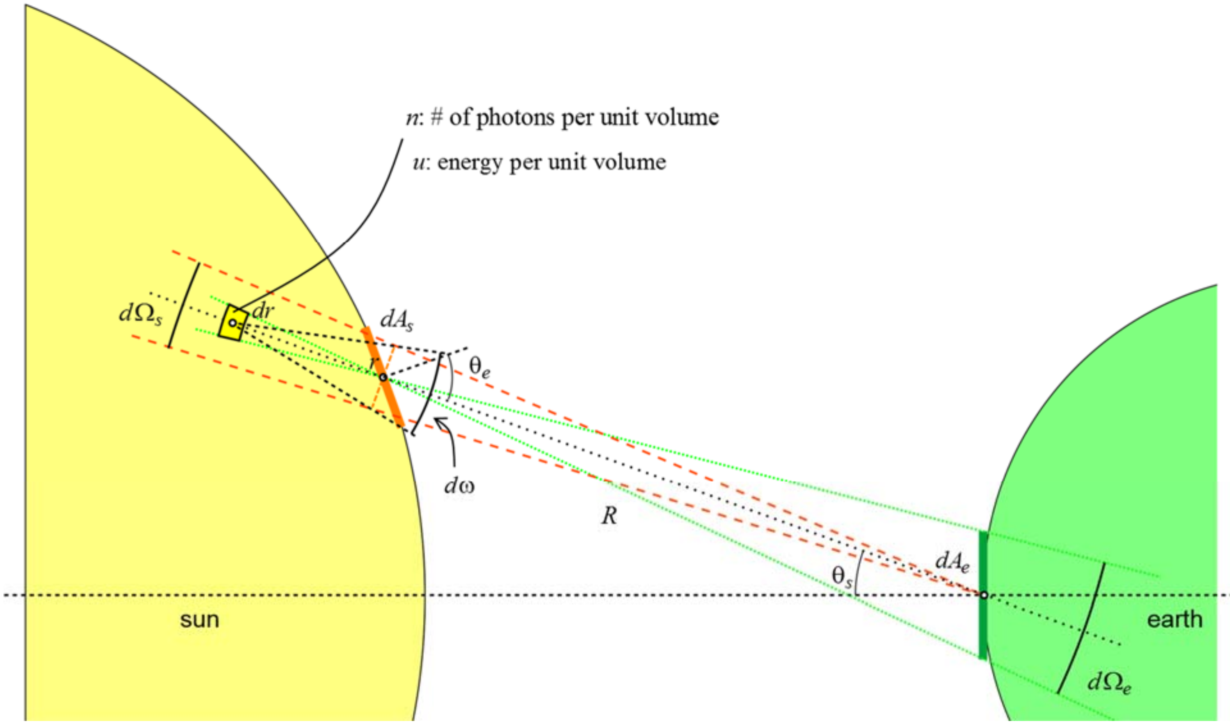
$$I_3 = \Gamma(4) \cdot \zeta(4) \approx (3!) \cdot \left(\frac{\pi^4}{90} \right) = \pi^4 / 15$$

which gives

$$u = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4 \approx \frac{(163.2) \cdot k^4 \cdot T^4}{h^3 c^3}$$

Blackbody radiation

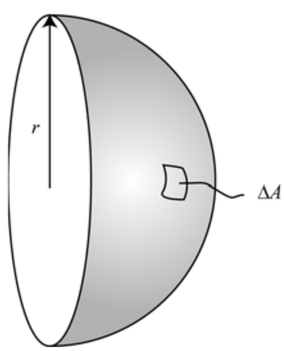
To find the terrestrial solar spectrum, we have to sort out several geometric factors related to the propagation of radiation from the interior of the sun to the earth's surface.



Blackbody radiation (II)

First we find the number of photons/photon energy/volume/solid angle within the interior of the photosphere. A differential solid angle is found specified by the differential area of a spherical section at a given radius

$$d\Omega = \frac{dA}{r^2}$$

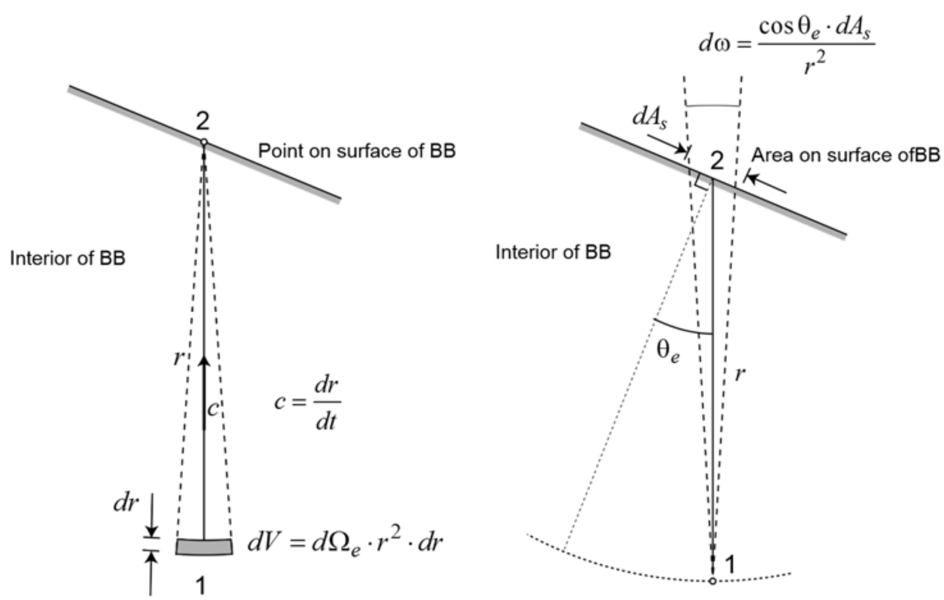


If radiation is emitted isotropically from the center of the sphere, the flux of photons per unit solid angle is $n/4\pi$ in units of $\#/(\mathrm{m}^3 \cdot \mathrm{sr})$.

Blackbody radiation (III)

Consider a differential volume element dV at point 1 within the sun at radius r from point 2 on the sun's surface, along a line connecting the earth to the sun, having differential radius dr and solid angle $d\Omega_e$.

We have $dV = d\Omega_e \cdot r^2 \cdot dr$. A portion of the radiation within this volume element at point 1 will approach point 2 at speed c . Light originating from point 1 radiates in all directions. Say the line connecting point 2 to a differential area dA_s containing point 2 makes an angle θ_e w.r.t the earth-sun line. Then dA_s presents a differential sold angle element of $d\omega = \cos\theta_e \cdot dA_s / r^2$.

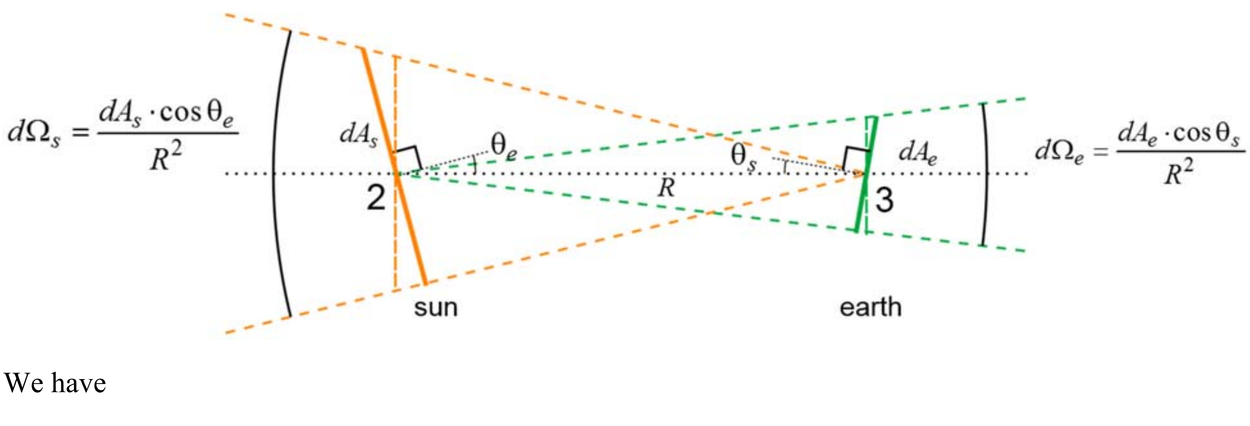


So the fraction of the light from point 1 reaching dA_s at point 2 is $d\omega/4\pi$. So the flux at point 2 is

$$n\cdot\frac{d\omega}{4\pi}\cdot\frac{dV}{dt}=\frac{n}{4\pi}\cdot\frac{\left(d\Omega_e\cdot \cancel{r^2}\cdot dr\right)\cdot\left(\dfrac{\cos\theta_e\cdot dA_s}{\cancel{r^2}}\right)}{dt}=\frac{n}{4\pi}\cdot\frac{dr}{dt}\cdot\cos\theta_e\cdot d\Omega_e\cdot dA_s$$

Blackbody radiation (IV)

Now consider the light emitted from the differential area dA_s at point 2 toward a point 3 on the surface of the earth at an angle θ_s from a line connecting the earth and sun centers.



We have

$$R^2 = \frac{dA_s \cdot \cos\theta_e}{d\Omega_s} = \frac{dA_e \cdot \cos\theta_s}{d\Omega_e}$$

So

$$\cos\theta_e \cdot d\Omega_e \cdot dA_s = \cos\theta_s \cdot d\Omega_s \cdot dA_e$$

Now we can substitute in the preceding factor

$$\frac{n}{4\pi} \cdot \frac{dV \cdot d\omega}{dt} = \frac{n \cdot c}{4\pi} \cdot \cos\theta_s \cdot d\Omega_s \cdot dA_e$$

Blackbody radiation (V)

We can resolve the spectral distribution in the preceding by

$$\frac{d}{dE}\left(\frac{n}{4\pi} \cdot \frac{dV \cdot d\omega}{dt}\right) = \frac{d}{dE}\left(\frac{n \cdot c}{4\pi} \cdot \cos\theta_s \cdot d\Omega_s \cdot dA_e\right) = \beta(E) \cdot \cos\theta_s \cdot d\Omega_s \cdot dA_e$$

using the spectral photon flux

$$\beta(E) = \frac{c}{4\pi} \cdot \frac{dn}{dE}, \; [\beta(E)] = \frac{\#}{\text{eV} \cdot \text{s} \cdot \text{m}^2 \cdot \text{sr}}$$

Integrating over solid angle gives

$$\int_{\Omega_s} \frac{n}{4\pi} \cdot \frac{dV \cdot d\omega}{dt} = \frac{n \cdot c}{4\pi} \cdot \left(\int_{\Omega_e} \cos\theta_s \cdot d\Omega_s\right) \cdot dA_e = \Phi \cdot dA_e$$

using the photon flux density

$$\Phi = \frac{n \cdot c}{4\pi} \cdot \int_{\Omega_s} \cos\theta_s \cdot d\Omega_s, \; [\Phi] = \frac{\#}{\text{s} \cdot \text{m}^2}$$

Alternatively, we can evaluate

$$\int_{\Omega} \frac{d}{dE}\left(\frac{n}{4\pi} \cdot \frac{dV \cdot d\omega}{dt}\right) = \frac{d\Phi}{dE} \cdot dA_e = \beta(E) \cdot \left(\int_{\Omega_s} \cos\theta_s \cdot d\Omega_s\right) \cdot dA_e = b(E) \cdot dA_e$$

where

$$b(E) = \frac{d\Phi}{dE} = \beta(E) \cdot \int_{\Omega_s} \cos\theta_s \cdot d\Omega_s, \; [b(E)] = \frac{\#}{\text{eV} \cdot \text{s} \cdot \text{m}^2}$$

is the spectral photon flux density.

Blackbody radiation (VI)

We have

$$b(E) = \beta(E) \cdot \int_{\Omega_s} \cos\theta_s \cdot d\Omega_s = F \cdot \beta(E)$$

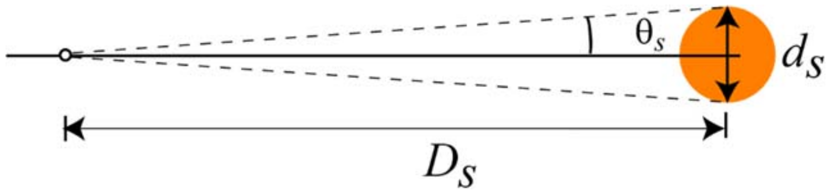
The geometric factor is

$$F = \int_{\Omega} \cos\theta \cdot d\Omega = \int_{\phi}\int_{\theta} \cos\theta \cdot \sin\theta \cdot d\phi \cdot d\theta, \; [F] = \text{sr}$$

Here we need to consider the range of integration for radiation from the sun incident on a point on the earth's surface. Using the reasonable assumption of a spherical sun:

$$F = \int_{\phi'=0}^{2\pi} d\phi' \cdot \int_{\theta'=0}^{\theta_s} \cos\theta' \cdot \sin\theta' \cdot d\theta' = \phi'\Big|_{\phi'=0}^{2\pi} \cdot \left(\frac{1}{2}\sin^2\theta'\right)\Bigg|_{\theta'=0}^{\theta_s} = \pi \cdot \sin^2\theta_s$$

where θ_s is the semi-angle subtended by the sun when viewed from earth.



The parameters are well know. The earth-sun distance is $D_s \approx 149.6\times10^9$ m and the sun's diameter is $d_s \approx 1.392\times10^9$ m . So $\tan\theta_s = (d_s/2)/D_s = 4.65\times10^{-3}$. Then $\theta_s \approx 4.6\,\text{mrad} = 0.267°$ si a reasonable estimate of the semi-angle of sun, viewed from earth.

Blackbody radiation (VII)
The geometric factor is maximized at the surface of the sun: $\theta_s^{(\max)} = 90°$, $F^{(\max)} = \pi$. At the surface of the earth: $F_s = \pi\cdot\sin^2\theta_s = F^{(\max)}\cdot f_s$, so $f_s = F/F^{(\max)} = \sin^2\theta_s \approx 2.2\times10^{-5}$. We can increase this factor by concentration to $f = X\cdot f_s$

$$X = \frac{f}{f_s} = \frac{\sin^2\theta}{\sin^2\theta_s}$$

The maximum concentration is therefore

$$X^{(\max)} = \frac{\sin^2(90°)}{\sin^2\theta_s} = \frac{1}{\sin^2\theta_s} = \frac{1}{f_s} = 4.6\times10^4$$

We could think of this ratio $f = X/X^{(\max)}$.

Blackbody radiation (VIII)
We often need to consider the radiated energy

$$\frac{u}{4\pi}\cdot\frac{dV\cdot d\omega}{dt} = \frac{u\cdot c}{4\pi}\cdot\cos\theta_s\cdot d\Omega_s\cdot dA_e$$

Then

$$\frac{d}{dE}\left(\frac{u}{4\pi}\cdot\frac{dV\cdot d\omega}{dt}\right) = E\cdot\beta(E)\cdot\cos\theta_s\cdot d\Omega_s\cdot dA_e$$

Here $E\cdot\beta(E)$ is the spectral energy flux. $[E\cdot\beta(E)] = \text{W}/(\text{eV}\cdot\text{m}^2\cdot\text{sr})$. We may also need

$$\int_{\Omega_s}\frac{u}{4\pi}\cdot\frac{dV\cdot d\omega}{dt} = \frac{u\cdot c}{4\pi}\cdot\left(\int_{\Omega_s}\cos\theta_s\cdot d\Omega_s\right)\cdot dA_e = P\cdot dA_e$$

where the power flux is

$$P = \frac{u\cdot c}{4\pi}\cdot\int_{\Omega_s}\cos\theta_s\cdot d\Omega_s\,,\ [P] = \frac{\text{W}}{\text{m}^2}$$

We may also need to evaluate

$$\int_{\Omega}\frac{d}{dE}\left(\frac{u}{4\pi}\cdot\frac{dV\cdot d\omega}{dt}\right) = L(E)\cdot dA_e$$

using the spectral irradiance

$$L(E) = E \cdot b(E)\,,\ [L(E)] = \frac{\mathrm{W}}{\mathrm{eV \cdot m^2}}$$

Blackbody radiation (IX)

The energy density in the sun is

$$u = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4$$

The power flux is

$$P = \frac{f \cdot c}{4} \cdot u = f \cdot \sigma_s \cdot T^4$$

where

$$\sigma_s = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67 \times 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m^2 \cdot K}}$$

is the Stefan constant. For the sun, assuming $T_s \approx 5760\,\mathrm{K}$, the power density at the sun's surface is found using $f = f^{(\max)} = 1$.

$$P^{(\max)} = \sigma_s \cdot T^4 \approx 62\ \frac{\mathrm{MW}}{\mathrm{m^2}}$$

On earth's surface, $f_s = 1 / X^{(\max)} = 1 / 4.6 \times 10^4$ gives

$$P_s = \frac{P^{(\max)}}{X^{(\max)}} \approx 1300\ \frac{\mathrm{W}}{\mathrm{m^2}}$$

Blackbody radiation (X)

For photovoltaics, we will need the spectral equivalent current flux:

$$j(E) = q \cdot b(E)\,,\ [j(E)] = \frac{\mathrm{A}}{\mathrm{eV \cdot m^2}}$$

The total equivalent current flux is:

$$J = q \cdot \Phi = q \cdot \frac{f \cdot c}{4} \cdot n\,,\ [J] = \frac{\mathrm{A}}{\mathrm{m^2}}$$

Notice that a flux density (or spectral flux density) is always proportional to a density (or spectral density) within the blacbody, e.g.:

$$b(E) = \frac{f \cdot c}{4} \cdot \frac{dn}{dE} \text{ and } \Phi = \frac{f \cdot c}{4} \cdot n$$

and

$$L(E) = \frac{f \cdot c}{4} \cdot \frac{du}{dE} \text{ and } P = \frac{f \cdot c}{4} \cdot u$$

Likewise

$$j(E) = q \cdot b(E) = q \cdot \frac{f \cdot c}{4} \cdot \frac{dn}{dE} \text{ and } J = q \cdot \frac{f \cdot c}{4} \cdot n$$

Blackbody radiation (XI)

We often need to consider a limited spectral range. Define

$$n(E_1,E_2)=\int_{E=E_1}^{E_2}\left(\frac{dn}{dE}\right)\cdot dE \text{ and } u(E_1,E_2)=\int_{E=E_1}^{E_2}\left(\frac{du}{dE}\right)\cdot dE$$

Then we have the following:

$$\Phi(E_1,E_2)=\int_{E=E_1}^{E_2} b(E)\cdot dE=\frac{f\cdot c}{4}\cdot n(E_1,E_2)$$

$$P(E_1,E_2)=\int_{E=E_1}^{E_2} L(E)\cdot dE=\frac{f\cdot c}{4}\cdot u(E_1,E_2)$$

$$J(E_1,E_2)=\int_{E=E_1}^{E_2} j(E)\cdot dE=q\cdot\frac{f\cdot c}{4}\cdot n(E_1,E_2)$$

Summary

Spectral densities and flux densities

Spectral photon density $\left(\frac{\#}{\mathrm{eV}\cdot\mathrm{m}^3}\right)$ $\frac{dn}{dE}=\frac{D(E)\cdot f(E)}{V}$	Spectral photon flux density $\left(\frac{\#}{\mathrm{eV}\cdot\mathrm{s}\cdot\mathrm{m}^2}\right)$ $b(E)=\frac{f_\omega\cdot c}{4}\cdot\frac{dn}{dE}$
Spectral energy density $\left(\frac{\mathrm{J}}{\mathrm{eV}\cdot\mathrm{m}^3}\right)$ $\frac{du}{dE}=E\cdot\frac{dn}{dE}$	Spectral irradiance $\left(\frac{\mathrm{W}}{\mathrm{eV}\cdot\mathrm{m}^2}\right)$ $L(E)=\frac{f_\omega\cdot c}{4}\cdot\frac{du}{dE}$

Densities and fluxes

Photon density $\left(\frac{\#}{\mathrm{m}^3}\right)$ $n(E_1,E_2)=\int_{E=E_1}^{E_2}\frac{dn}{dE}\cdot dE$	Photon flux density $\left(\frac{\#}{\mathrm{s}\cdot\mathrm{m}^2}\right)$ $\Phi(E_1,E_2)=\frac{f_\omega\cdot c}{4}\cdot n(E_1,E_2)$
Energy density $\left(\frac{\mathrm{J}}{\mathrm{m}^3}\right)$ $u(E_1,E_2)=\int_{E=E_1}^{E_2}\frac{du}{dE}\cdot dE$	Power flux density $\left(\frac{\mathrm{W}}{\mathrm{m}^2}\right)$ $p(E_1,E_2)=\frac{f_\omega\cdot c}{4}\cdot u(E_1,E_2)$

Wavelength representations

We often wish to resolve the spectra in terms of wavelength, rather than energy. We will use an E superscript for an energy resolved quantity and a λ superscript for a wavelength resolved quantity. For example

$$b^{(E)}=\frac{f\cdot c}{4}\cdot\left|\frac{dn}{dE}\right| \text{ and } b^{(\lambda)}=\frac{f\cdot c}{4}\cdot\left|\frac{dn}{d\lambda}\right|$$

or

$$L^{(E)}=\frac{f\cdot c}{4}\cdot\left|\frac{du}{dE}\right| \text{ and } L^{(\lambda)}=\frac{f\cdot c}{4}\cdot\left|\frac{du}{d\lambda}\right|$$

to relate these we note that

$$\frac{dn}{d\lambda} = \frac{dE}{d\lambda} \cdot \frac{dn}{dE} \text{ and } \frac{du}{d\lambda} = \frac{dE}{d\lambda} \cdot \frac{du}{dE}$$

The photon energy is $E = hc/\lambda$. So

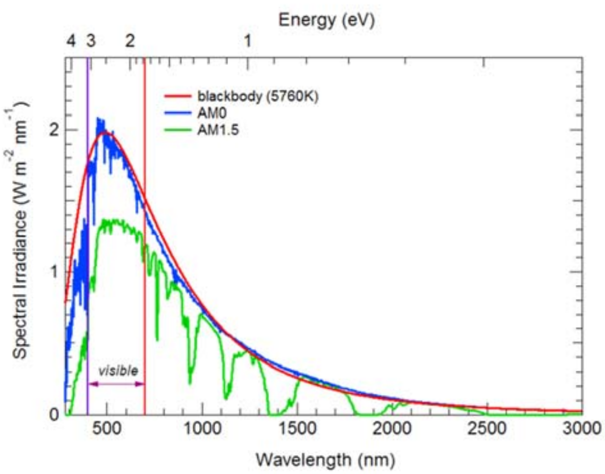
$$dE = -\frac{hc}{\lambda^2} = -\frac{E^2}{hc}$$

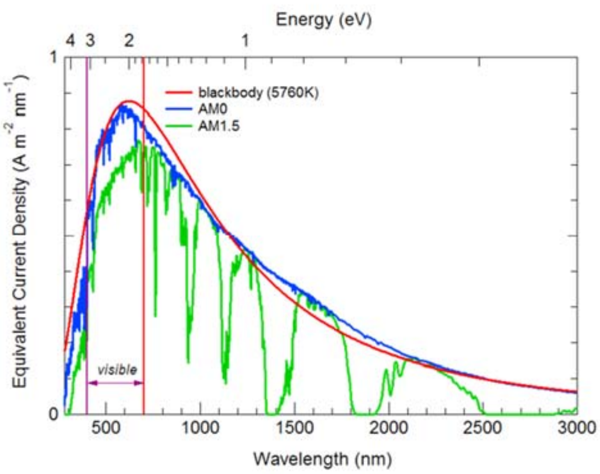
Now we have

$$b^{(\lambda)} = \frac{hc}{\lambda^2} \cdot b^{(E)} \text{ and } L^{(\lambda)} = \frac{hc}{\lambda^2} \cdot L^{(E)}$$

Real vs. calculated solar spectra

We can use the above conversion to calculate the wavelength-resolved blackbody spectral irradiance or equivalent current density for the blackbody. Comparison to experimental solar spectra are shown below.





Note that AM refers to air mass, or more specifically, the angle of the sub from vertucal. Say the sun as at an angle γ_S w.r.t the horizon. If the thickness of the atmosphere directly overhead is h_1 , then

$$\sin\gamma_s = h_1/h$$

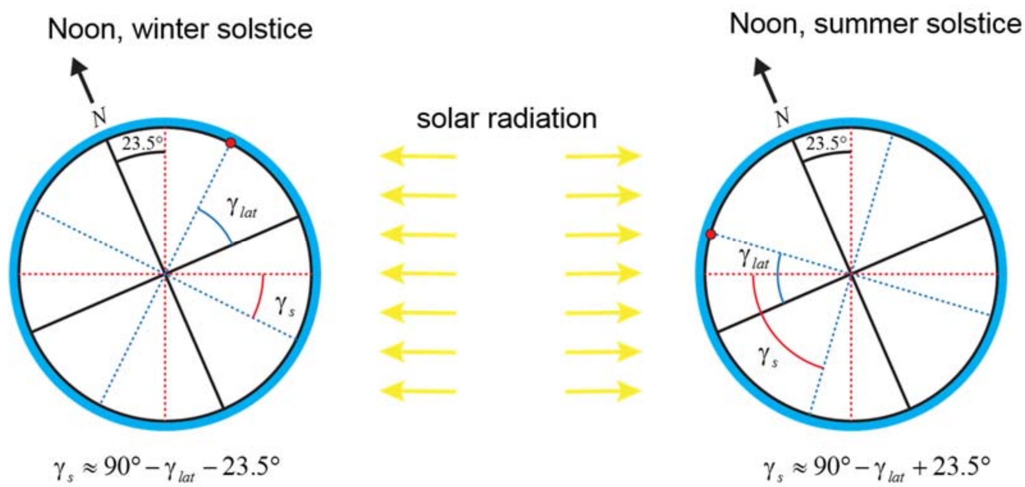
where h is the thickness of atmosphere between the earth's surface and the sun. Air mass is defined as

$$n_{\text{air mass}} = \frac{1}{\sin\gamma_s}$$

So AM1.5 corresponds to a solar angle of $\gamma_s \approx 42°$.

Angle of sun from horizon

The angle of the sun from the horizon is related to the time of day, of course, and the latitude, as well as the tilt of the earth's rotation axis from its plane of revolution. The rotation axis is at 23.5° from the normal to the plane of revolution. The cases of (solar) noon at the winter and summer solstice in the northern hemisphere are shown below



We can say that

$$\gamma_s \approx (90° - \gamma_{lat}) + (23.5°) \cdot \cos\left[2\pi \cdot \left(\frac{N}{365 \text{ days}}\right)\right]$$

where N is the number of days since the summer solstice.